Journal of Statistical Physics, Vol. 54, Nos. 1/2, 1989

Phase Transitions in Systems with a Finite Number of Dominant Ground States

J. Bricmont[1,2] and J. Slawny[3]

Received February 8, 1988; revision received August 8, 1988

We develop a theory of low-temperature phases of discrete lattice systems which is guided by formal perturbation theory, and which in turn yields its rigorous justification. The theory applies to many systems with an infinite number of ground states for which the perturbation theory predicts a finite number of low-temperature phases. We illustrate it on a number of examples.

KEY WORDS: Phase transitions; low temperatures; Pirogov–Sinai theory; perturbation theory; dominant ground states.

INTRODUCTION

We develop here a theory of low-temperature phases of discrete lattice systems which is guided by the formal perturbation theory,[12,32] and which in turn yields its rigorous justification. The theory applies to a large class of models which have a standard low-temperature expansion (*regular systems*). This class includes most systems with a finite number of periodic ground states, but also many systems with an infinite number of ground states for which, however, the perturbation theory predicts a finite number of low-temperature phases. We illustrate the theory on the examples of the Balanced Model, the Ising antiferromagnet on the face-centered cubic lattice, the three-dimensional ANNNI model (whose phase diagram was first calculated using perturbation theory by Fisher and Selke[12] and was rigorously derived in refs. 8–10 and 25), and an antiferromagnet on the

[1] Physics Department, Princeton University, Princeton, New Jersey 08540.
[2] Permanent address: Institut de Physique Théorique, B-1348 Louvain-la-Neuve, Belgium.
[3] Center for Transport Theory and Mathematical Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061.

0022-4715/89/0100-0089$06.00/0

simple cubic lattice with next nearest neighbor interaction and magnetic field.

We shall vindicate the predictions of perturbation theory under somewhat stronger assumptions than regularity. (Actually, we are not aware of any regular model for which our assumptions are not satisfied.) Some of the models discussed here have already been analyzed rigorously in refs. 8–10 and 25. However, our method differs from theirs and we regard our main results as being the general Theorems A and B (Sections 1 and 2.5) rather than the analysis of the various examples. We also have some results on nonregular systems, e.g., the Heilmann–Lieb models of liquid crystals;[19] these are mentioned in Section 5 and will be the subject of a separate publication.

The technically most demanding part of the theory is essentially contained in the work of Bricmont *et al.* (BKL),[3] which itself is an extension of the Pigorov–Sinai theory[29,30] (see also ref. 46 for a related extension). Here the main difference with refs. 29 and 30 is that the low-temperature phases are obtained not as perturbations of the corresponding ground states, but of states obtained by including the low-energy excitations of the ground states with the lowest perturbative free energy (the *dominant* ground states)—a special case of expanding about restricted ensembles of BKL. When the number of the dominant ground states is finite, which is our basic assumption, one has the picture of contours separating regions belonging to the different restricted ensembles, as in the Pirogov–Sinai theory. However, these contours are defined here on a temperature-dependent scale ("renormalized")—this is another new ingredient of the theory. We also develop a technique which allows one to prove our version of the "Peierls condition." We shall now review the Pigorov–Sinai theory, then explain how the perturbation theory works and how one justifies it. For a more detailed introduction to these matters, see ref. 4.

Pirogov and Sinai developed a rigorous theory of first-order phase transitions.[29,30] Their theory, combined with the results of ref. 38, gives a complete description of low-temperature phase diagrams of discrete lattice systems with a finite number of (periodic) ground states. At each site of a lattice of dimension two or larger, one has a "spin" variable assuming a finite number of values; the Hamiltonians are of finite range. The basic Hamiltonian H_0 has a finite number of periodic ground states and one considers a family of perturbations

$$H_\mu = H_0 + \sum_i \mu_i H_i$$

of H_0 removing completely the degeneracy of the ground states. Furthermore, H_0 is required to satisfy the "Peierls condition": inserting one ground

state in a volume Λ into another ground state costs an energy proportional to the boundary of Λ. This condition is common to most versions of the "Peierls argument."[29] The main result of ref. 29 is that, when the Peierls condition is satisfied, at low temperatures the phase diagram is a small deformation of the zero-temperature one.

Several extensions and alternative versions of this theory have appeared: extensions to continuous systems such as field theories,[44] Widom–Rowlinson models (BKL), lattice models with continuous spin,[43] infinite-range interactions,[27] complex Hamiltonians,[1,16,28] and to some systems with an infinite number of periodic ground states.[1,8–10,25]

We define now *regular systems*. Consider a Hamiltonian H_0; an *excitation* is a configuration which coincides with a ground state of H_0 outside of a finite set of sites. If the energy of an excitation (relative to the corresponding ground state) tends to infinity with the size of the region where it differs from the ground state, then H_0 and the corresponding model are called *regular*. Clearly, all models satisfying the Peierls condition are regular, since then the energy of an excitation is of order of the size of the boundary of its support. In this definition and in most of the discussion of this Introduction we consider only periodic, globally defined ground states. However, as we mention later, our work involves ground states which are defined only locally, some of which may not extend to ground states of the entire lattice.

A simple model which is regular but has an infinite number of ground states and does not satisfy the Peierls condition is the Balanced Model of ref. 14. This is chosen here as the basic illustrative example. We consider its three-dimensional version, on the simple cubic lattice $\mathbb{Z}^3$. The interaction consists of a nearest neighbor (n.n.) ferromagnetic part and a next nearest neighbor (n.n.n.) antiferromagnetic part:

$$H_0 = -2 \sum_{\text{n.n.}} \sigma_a \sigma_b + \frac{1}{2} \sum_{\text{n.n.n.}} \sigma_a \sigma_b \tag{0.1}$$

We imbed H_0 into a one-parameter family H_μ of Hamiltonians:

$$H_\mu = H_0 + \mu \frac{1}{2} \sum_{\text{n.n.n.}} \sigma_a \sigma_b \tag{0.2}$$

and study the system for small μ.

For μ negative there are two ground states, one equal to $+1$ and the other to -1 everywhere. These ground states are equivalent, i.e., related by a symmetry of the Hamiltonian. For positive μ there are six equivalent ground states which consist of alternating plus and minus planes perpendicular to one of the three coordinate axes. For $\mu = 0$, i.e., for the

Hamiltonian (0.1), the family of ground states is infinite, but of zero entropy. Apart from the above eight ground states, there are ground states of lesser symmetry. Each of them is obtained from the plus ground state by picking one of the three coordinate axes and fliping all spins in a number of planes perpendicular to the axis (*layered* structure of the ground states). Note that flipping the spins in a plane is not a symmetry of the Hamiltonian, so that the ground states of this model are not equivalent in general.

To see that this model is regular but fails to satisfy the Peierls condition, consider the excitation of the plus ground state obtained from it by flipping all spins inside a cube of edge length L. It is easy to see that the energy of this excitation is of order L, when $L \to \infty$, which ensures the regularity, while the Peierls condition would require it to be of order L^2. Indeed, the faces of the cube do not contribute to the energy, since they locally coincide with a ground-state configuration, but the edges of the cube do. A similar argument shows that the two-dimensional version of this model is not even regular. This lack of regularity also occurs in the two-dimensional ANNNI model, or in the one-dimensional Ising model.

Next we recall the formal perturbation theory for the phase diagram of regular systems. The basic notion here is that of a *dominant ground state.* Since the ground states are not, in general, related by symmetry, different ground states have different excitations. Every excitation can be decomposed into connected components, called *elementary excitations.* The ground states which have the largest number of elementary excitations of lowest energy are called *dominant.* We will now illustrate these notions on the example of the Balanced Model.

Excitations of (0.1) up to order 3 are pictured on Fig. 6. The first-order excitations, energy $E_1 = 12$, are obtained by flipping one spin of an arbitrary sign; the second-order excitations are found by flipping a pair of nearest neighbors of the same sign, plus or minus; the third-order excitations are found by flipping three or four spins of the same sign in configurations X_{31} and X_{32}. It is clear that all the ground states have the same number of excitations of the first order. It is also obvious that in the next order the plus and minus ground states have more excitations, i.e., that these ground states are dominant in order 2: if a ground state contains a pair of neighboring planes of opposite sign,then there are no second-order excitations coming from flipping of the intraplane pairs of nearest neighbors.

The first prediction of the formal perturbation theory is that at low temperatures the periodic Gibbs states correspond to small perturbations of the dominant ground states, and only of those. Thus, for the Balanced Model there should be only two periodic Gibbs states, at low temperatures,

each of them having as typical configurations small perturbations of the corresponding dominant ground state. That this is indeed the case follows from Theorem A of the beginning of Section 1.

In general, unless there is some accidentally large symmetry of H_0, as in ferromagnetic systems, where all the ground states are related by symmetries of the Hamiltonian,[32,37,41] the set of dominant ground states will be finite. In fact, it is in general unique except for some symmetries of H_0, which tend to form a finite group (global spin flip, as in the balanced model, or lattice rotations and translations over the period of the ground states).

One can also imbed the Hamiltonian H_0 in a family H_μ of Hamiltonians, as in the example (0.2), and ask for the phase diagram in the μ space for low temperatures and μ small.

To answer this question, one proceeds as follows: fix a cutoff energy E, and for each ground state G consider the set of its elementary excitations whose energy (relative to G) does not exceed E. Because of the regularity assumption, this can be viewed as the phase space of a gas of "particles" with a finite number of species, the different elementary excitations, which interact via a hard-core exclusion forbidding their overlapping. One can define the free energy $f_E(\mu \mid G)$ of this gas, for any small μ, which has standard small-activity expansion.[32] This holds for any E, although the radius of convergence may depend on E. Note that this free energy, unlike the free energy of the complete system, will depend in general on G, since the phase space considered depends very much on G. To find the phase diagram for μ small, one compares the cutoff free energies $f_E(\mu \mid G)$ of different ground states G. The dominant ground states are those which have the lowest cutoff free energy, as the temperature goes to zero, for some E (and hence for all $E' > E$). Again the dominant ground states should give rise to the Gibbs states of H_μ for small temperature (and μ).

Let C-ground state mean a plus or minus ground state, and D-ground state mean any ground state with alternating plus and minus planes; the results of this analysis (see Section 4.2) for the perturbation (0.2) of the Balanced Model yield first a curve $\beta \mapsto \mu_{(3)}^{C,D}(\beta)$,

$$\beta \mu_{(3)}^{C,D}(\beta) = \tfrac{1}{2} e^{-\beta E_2} + e^{-\beta E_3}$$

obtained from setting equal the cutoff free energy of the two C-ground states to the cutoff free energy of any of the six D-ground states. On $\mu_{(3)}^{C,D}(\cdot)$ the eight ground states are dominant in order 3. Then our main result (Theorem B) asserts that there is a curve $\mu^{C,D}(\cdot)$ with $\mu^{C,D}(\beta) - \mu_{(3)}^{C,D}(\beta) = o(e^{-\beta E_3})$ such that for $\mu < \mu^{C,D}(\beta)$ one has only the two C-phases, for

$\mu > \mu^{C,D}(\beta)$ one has the six D-phases, and on the curve $\mu(\cdot)$ one has eight pure phases. Of course, the phase diagram may be much richer, as the three-dimensional ANNNI models demonstrates.[12]

This perturbative analysis can be completely justified using the Pirogov–Sinai theory, when there is a finite number of periodic ground states and the Peierls condition holds.[32]

To explain the main difficulty encountered in models with an infinite number of ground states, which tend to be at best regular but do not satisfy the Peierls condition, we consider again the Balanced Model. To show that the two dominant ground states lead to different Gibbs states at low temperatures, let us insert again a large cube of the minus ground state into the plus ground state. As we saw, the faces do not cost any energy, because the configuration coincides there with a ground state except on their boundary, i.e., on the edges of the cube. However, this ground state is not a dominant one, since it contains two adjacent planes of plus spins and minus spins. Along these faces one has fewer low-energy excitations than in the plus or the minus ground state. Hence, these faces, although not costing any energy, cost a free energy, because of the loss of low energy excitations they cause. However, this free energy is very small at low temperatures, being of order $e^{-\beta E_D}$, where D is the order in which the plus and the minus ground states dominate (here $D=2$ and $E_D=16$). One has to show now that this small factor is sufficient for the Peierls argument to work, and to prove that the plus and the minus ground states yield different phases.

Our main idea is to define new, renormalized, contours, so that the regions of the lattice where the configuration coincides with a nondominant ground state are included in the contours. If the family of the dominant ground states is finite, the new contours will have all the geometrical properties of the usual contours, since the faces discussed above will be part of the contour. Of course, the first thing to do, in order to implement this idea, is to define the contour, not with respect to the ground state, but with respect to the ground states and their low-energy excitations. We introduce, following BKL, restricted ensembles: They are simply the gases of low-energy excitations that we have associated with the ground states. They are restricted because we allow only a certain, ground-state-dependent, set of configurations. They are also dilute in the sense of BKL, because there is a convergent (low-temperature) expansion in these ensembles, which, in a gas language, is a low-fugacity expansion. In particular, one has a good control on the free energy in a finite volume: one can separate the bulk term from the boundary one, and get estimates on the latter.

The new contours will be the regions of the lattice where the configuration does not belong to a restricted ensemble of a dominant ground

state. Since the restricted ensemble of the nondominant ground states have higher free energy, these contours will have some damping factor. However, this factor will be exponentially small in β, because the difference in free energy is only due to low-energy excitations. To compensate for this, we shall define the contour on a temperature-dependent scale. Namely, cover the lattice with large cubes of volume $e^{\beta E}$ with $E > E_D$; define a cube C to be irregular, for a given configuration, if the configuration restricted to C either belongs to the restricted ensemble of a nondominant ground state (cube of type I) or contains an excitation (of any ground state) of energy E' strictly larger than E_D (cube of type II), where E' is chosen in a suitable way.

To see that irregular cubes are highly unlikely, one first makes the observation that in all our examples any cube of type I contains a region of size $|C|^{2/3}$ in which the local ground state is nondominant (this is abstracted as our condition (Q) of Section 3.3; an exponent smaller than 2/3 would also go), and therefore each cube of type I costs a free energy

$$|C|^{2/3} e^{\beta E_D} = e^{\beta(2E/3 - E_D)}$$

where is large if $\frac{2}{3}E > E_D$ (actually, the precise estimates are somewhat more delicate; see Section 3.3). For the cubes of type II one has the following rough estimates of the probability of finding even one excitation of energy E' in a box C: summing over all families of excitations in C, one obtains

$$\sum_n (e^{-\beta E'})^n \binom{|C|}{n} = (1 + e^{-\beta E'})^{|C|} - 1 \cong |C|\, e^{-\beta E'} \cong e^{-\beta(E' - E)}$$

which is again very small for β large if $E' > E$. The basic fact that we use here is that the set of excitations is discrete, so that different excitations are associated with different length scales: the size of the cube in which they have a significant probability to appear. This size is of order $e^{\beta E}$ for an excitation of energy E.

One final observation: whatever the scale on which the new contours are defined, the number of them containing a given number n of cubes is of order c^n, with c independent of the scale (provided that the number of cubes overlapping with a given cube is fixed). Combining these remarks, one easily obtains a Peierls bound, which proves the coexistence of the plus and minus phases, and only of those, at low temperatures; see ref. 4 for an elementary version of this argument.

One also wants to analyze what happens for small perturbations of the original Hamiltonian, e.g., for (0.2). With the notions of restricted ensemble and of large-scale contours, one can rather easily adapt the Pirogov–Sinai

method to this situation. Indeed, the new contours satisfy the essential property of their contours, namely the Peierls condition, with the difference that disjoint contours are no longer independent, because in their complement one no longer has a ground state, but a restricted ensemble of a (dominant) ground state, which induces interactions between (outer) contours. This situation is common to many extensions of Pirogov–Sinai theory, e.g., to field theory[44] or to continuum fluids,[3] where very low-energy excitations around a ground state cannot be put into the contours. Here we use the method of ref. 3 to treat these interacting contour models.

The contours of the Balanced Model obtained by inserting a cube of minus spins into the plus ground state have a number of simplifying features not present in the case of more general and complicated contours. In fact, for all these arguments to go through we need a more precise ("localized") version of various assumptions. These assumptions do not surface in the above discussion or in the usual perturbative calculations, which involve only global (and periodic) ground states. We note that domination in the local form implies the existence of surface tension between the coexisting phases (Section 5.7), a result which requires an independent calculation when only the periodic ground states are used. We explain now our technical assumptions.

(a) *Retouch property.* It means that a configuration which is "excited" in a small region of the lattice surrounded by a region with the lowest possible energy can be changed ("retouched") in this region in a unique way so that the energy has the lowest possible value everywhere (Fig. 2). This property does not hold in the one-dimensional Ising model (where one may have to flip a half-line of spins to remove an isolated excitation) and in several nonregular models (two-dimensional ANNNI, two-dimensional balanced). It holds in all the regular models we have considered. It is similar to the decomposition property, a notion appearing in the study of ferromagnetic models,[20,26] in which case it is equivalent to the regularity. It implies that the restricted ensemble of low-energy excitations forms a gas with a finite number of "species of particles" with small activities when the temperature is low, and *interacting through exclusion only*, and therefore has a standard low-temperature (= small-activity) expansion. The last property was obvious in the models of BKL; here it is formalized as the retouch property, which itself follows from a detailed analysis of the geometry of the ground states and is related to their *layered* structure.

(b) *Domination in its local form.* As mentioned earlier, the notion of domination discussed above involves periodic ground states only and is define only in the thermodynamic limit, while we need to do finite-volume

estimates on partition functions. In particular, one has to deal with local ground states which do not extend to ground states defined on all the lattice. Thus, we shall demand that the domination can be suitably expressed in finite volume, what we call the "local" form of the domination. In all models we consider, this is best exhibited through corresponding properties of an *effective potential* of the gas of low-energy excitations. [When no perturbations are considered—$\mu = 0$ in (0.2)—one can usually infer the domination by inspection.]

(c) *Condition* ($\mathscr{L}$). This we consider to be the least satisfactory of our assumptions. It is somehow related to the fact that the number of ground states is not too large. In the models we consider it is implied by the fact that the ground states have layered structure: all the ground states can be obtained from a finite number of them by flipping all the spins in a number of planes (layers), not lines.

As a rule, we find that our "technical" hypotheses are not more restrictive than the condition of regularity. However, they do make the proofs somewhat subtle and longer.

We give now an outline of the paper. After introducing in Section 1 the framework, we consider in Theorem A the simplest version of our results, namely the situation when all the ground states are equivalent and when we go up, i.e., raise the temperature, starting from zero, without introducing any perturbations of H_0. Then we introduce several regular models with an infinite number of ground states, describe their global ground states and low-energy excitations, deduce which of the ground states are dominant, and state explicitly the consequences of Theorem A for the models. Then, in Section 2 we introduce our main technical tools: the restricted ensembles of low-energy excitations of the ground states and the effective potentials. To define them, we need to introduce the retouch property. Using these notions, we state the precise form of local domination that we need. Section 3 is devoted to the proofs of the general Theorems A and B. We return to the examples in Section 4, where we give proofs of the retouch property and local domination. Section 5 contains assorted remarks.

1. THE SETUP, EXAMPLES, AND PRELIMINARY RESULTS

After introducing the framework, we formulate our main result in its simplest form (Theorem A). Then we describe the global ground states of the models and the excitations which yield the domination, and apply to the models Theorem A. A proof of a more precise (localized) version of

domination results and of other properties [retouch, Condition $(\mathscr{L})$], which form the basis of our analysis, is given in Section 4.

Our lattice $\mathbb{L}$ is a discrete $\mathbb{Z}^3$-invariant subset of $\mathbb{R}^3$, and the configuration space $\mathscr{X}$ is $S^{\mathbb{L}}$, where S is a finite set, or $\mathscr{X}_\Lambda = S^\Lambda$ if one considers a system on a subset Λ of the lattice. Most of our examples are formulated in spin language. In the case of spin 1/2 the configuration space is $\mathscr{X} = \{-1, 1\}^{\mathbb{L}}$; σ_a (or s_a) is the usual spin variable at a lattice point a. The Hamiltonian is written as

$$H = \sum_B \Phi_B \tag{1.1}$$

where B runs over a family of finite subsets of the lattice, called *bonds*.

We will consider symmetries of the system induced by transformations of the lattice, mostly translations and rotations, and by pointwise transformations, like spin flips. We adopt the usual definitions (ref. 30; ref. 32, Section I.A.6) of symmetries and states of the system, in particular that of the equivalence of configurations under symmetries of the Hamiltonian. We refer to refs. 30 and 32 for definitions of Gibbs states and equilibrium states. By pure phase we understand an extremal equilibrium state. The low-temperature pure phases investigated here are obtained as limits of finite-volume Gibbs states with boundary conditions defined by the dominant ground states. As usual, correlations in these states decay exponentially.

We give now a simple version of our results, when all the dominant ground states are equivalent and no perturbations of the original system are introduced. The retouch property and Condition $(\mathscr{L})$ are discussed in the Introduction, and again in Sections 2.2 and 3.3, where they are defined in a more precise way, and finally are proved to hold for each of the models in Section 4.

Theorem A. Suppose that H_0 has the retouch property, i.e., that the low-temperature excitations of H_0 form a "gas" interacting through exclusion only, that the Condition $(\mathscr{L})$ is satisfied, and that there is a finite family $\mathscr{G}^*$ of dominant ground states, all equivalent under symmetries of H_0. Then for low enough temperatures there are exactly $|\mathscr{G}^*|$ different pure phases,[4] each of them being a small perturbation of the corresponding ground state.

This is proved in Section 3.5.

We introduce now a few models which will serve as an illustration of the general scheme, and which are also of independent interest.

[4] For any set S, $|S|$ denotes its number of elements.

1.1. The Balanced Model

This is the simplest model with an infinite number of ground states which illustrates the theory. It has been introduced and discussed extensively in ref. 14, Section 8. The three-dimensional version of the model is discussed in the Introduction. In $\mathbb{Z}^d$ it takes the form

$$H_0 = -(d-1)\sum_{\text{n.n.}} \sigma_a\sigma_b + \frac{1}{2}\sum_{\text{n.n.n.}} \sigma_a\sigma_b \tag{1.2}$$

where the first sum extends over (unordered) pairs of nearest neighbors (n.n.) and the second over pairs of next nearest neighbors (n.n.n.). In ref. 14 a number of hypotheses regarding the model are advanced and the reflection positivity method is used to prove it has a long-range order in dimension 5 and larger. We refine the conjectures of ref. 14, and applying Theorem A to the model (1.2), we obtain that if $d \geqslant 3$, it has exactly two phases at low temperatures. Furthermore, if one adds to (1.2) a n.n.n. perturbation [see (0.2)], one obtains the phase diagram of Fig. 7, as described in the Introduction and proved in Section 4.1.

1.2. Spin-1/2 Antiferromagnet on a Face-Centered Cubic Lattice

The lattice $\mathbb{L}$ is the face-centered cubic (fcc) lattice, whose elementary cell is pictured in Fig. 8. The Hamiltonian is

$$H_0 = \sum_{\text{n.n.}} \sigma_a\sigma_b \tag{1.3}$$

More generally, one considers a two-parameter family $H(\alpha, h)$ of perturbations of (1.3):

$$H = H(\alpha, h) = \sum_{\text{n.n.}} \sigma_a\sigma_b + \alpha \sum_{\text{n.n.n.}} \sigma_a\sigma_b - h\sum_a \sigma_a \tag{1.4}$$

Every point has 12 n.n., one at the center of each plaquette incident upon it, and six n.n.n.—the n.n. on its $\mathbb{Z}^3$ orbit.

The ground states of the model (1.3) which are of interest here can be described as follows (ref. 6; see Section 4.2). There is a family $\mathscr{G}^*$ of six most symmetric ground states with alternating planes of pluses and minuses, planes perpendicular to one of the three coordinate axes. The ground states of $\mathscr{G}^*$ are all *equivalent*, i.e., related by symmetries of the Hamiltonian (translations and rotations). To obtain the other ground states, choose one of the three coordinate axes and then one of the two

ground states, with planes perpendicular to this axis. Furthermore, select any of the two remaining coordinates axes and any number of planes perpendicular to it. Finaly, flip all the spins in the selected planes.

Figure 9 gives low-energy excitations of the model. The ground states of $\mathcal{G}^*$ are dominant in order three: The first-order excitations are obtained by flipping one spin, positive or negative; the second order by flipping a pair of n.n. spins of opposite signs. There is no difference between the ground states as far as the number of excitations of these types is concerned. The same refers to all third-order excitations apart from X_{36}. This last type of excitation, obtained by flipping four spins marked by f's on Fig. 8, favors the six most symmetric ground states. As is shown in Section 4.2, by Theorem A this implies that at low but nonzero temperatures there are precisely six pure phases—small perturbations of the corresponding ground states of $\mathcal{G}^*$. In Section 4.2 we give results for $\alpha, h \neq 0$.

1.3. The ANNNI Model[8–10,12,25]

For the sake of notational simplicity we consider the three-dimensional case only. Our results hold in higher dimensions, too. The two-dimensional model is not regular any more and description of its low-temperature properties in a rigorous way is an open problem, presumably harder than the one we solve here.[13,36]

In the notation of ref. 10, the Hamiltonian is

$$H = -J_0 \sum_{\text{hor n.n.}} s_a s_b - J_1 \sum_{\text{ver n.n.}} s_a s_b - J_2 \sum_{\text{ver n.n.n.}} s_a s_b \tag{1.5}$$

where s takes values ± 1, the first sum is over all pairs of horizontal nearest neighbors, i.e., n.n. pairs joined by a line perpendicular to the z axis; the second and the third sums are over vertical pairs of nearest and next nearest neighbors, respectively. The coupling constants J_0 and J_1 are positive and J_2 negative.

Due to the ferromagnetic (horizontal) part of the interaction, the ground states of (1.5) consist of horizontal layers of plus and minus spins. The arrangement of these layers is described by ground states of a one-dimensional system with interaction given by the vertical part of (1.5)—this is the stacking construction of Section 4.4. Thus, it is enough to describe the ground states of the corresponding one-dimensional model.

The structure of these ground states is determined by a competition between the forromagnetic part of the interaction (coupling constant J_1) and the antiferromagnetic part (coupling constant J_2). Let $\kappa = -J_2/J_1$. Then for $\kappa > 1/2$ any configuration with no plus or minus spins surrounded

by spins of opposite signs is a ground state. This family of ground states has a distinguished subfamily of periodic ground states which, according to the perturbative calculations of ref. 12, should yield the low-temperature phases:

There are two ground states of type $\langle \infty \rangle$, the ferromagnetic plus and minus ground states; six of type $\langle 3 \rangle$, ground states with alternating stripes of width 3 of plus and minus planes; four of type $\langle 2 \rangle$; and $2(2j+3)$ of type $\langle 2^j 3 \rangle$, $j = 1, 2, 3, \ldots$, which will not be considered here. The ground states of each type are equivalent under translations and global flip symmetries of the Hamiltonian.

For $\kappa = 1/2$ and J_0 large enough the ground states of type $\langle 3 \rangle$ are dominant in first order, which can be easily understood as follows. For J_0 large enough the lowest order excitations are obtained by flipping one spin. The energy of such an excitation is $-8J_0$ (from the interplane interaction) plus a term coming from the corresponding one-dimensional system. The last term is zero if the configuration obtained by flipping the spin is a ground state again. Now, a $\langle 3 \rangle$ ground state of the one-dimensional system consists of alternating triples of plus and minus spins. Flipping any border spin of such a configuration (i.e., a spin having a n.n. of opposite sign) leads to a ground state again (no isolated plus or minus spins). And it is clear that no other ground state will have as many lowest energy excitations as the ground states of type $\langle 3 \rangle$ do. Hence, by Theorem A for $\kappa = 1/2$ the model (1.4) has exactly six pure phases at low temperatures. Results for $\kappa \neq 1/2$ are discussed in Section 4.3.

1.4. The Stacked Antiferromagnet[4]

This is a model on the simple cubic lattice $\mathbb{Z}^3$ with the Hamiltonian

$$H = \sum_{\text{hor n.n.}} \sigma_a \sigma_b - \alpha \sum_{\text{hor n.n.n.}} \sigma_a \sigma_b - h \sum_a \sigma_a + J \sum_{\text{ver n.n.}} \sigma_a \sigma_b \tag{1.6}$$

The notation is similiar to that of (1.5), with the difference that here one stacks two-dimensional systems, each living on a plane perpendicular to the z axis. For $J > 0$, the case considered here, the ground states of (1.6) are obtained by repeating in each of the planes a ground state of an antiferromagnetic system on the simple square lattice $\mathbb{Z}^2$, with next nearest neighbor interactions in an external magnetic field, i.e., with Hamiltonian

$$H = \sum_{\text{n.n.}} \sigma_a \sigma_b - \alpha \sum_{\text{n.n.n.}} \sigma_a \sigma_b - h \sum_a \sigma_a \tag{1.7}$$

These ground states have been described in Section 5 of ref. 4 and are obtained by patching the plaquettes of Fig. 1 here.

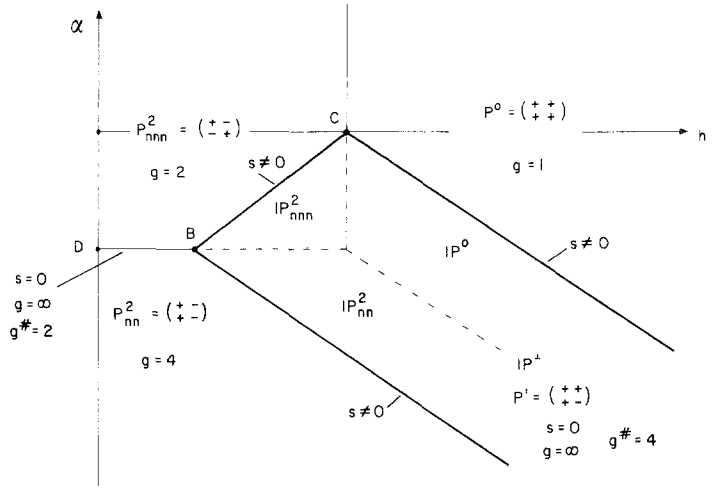

Fig. 1. Ground states of the model (1.7) are obtained by patching the indicated configurations of plaquettes. The ground-state entropy s is nonzero on the bold lines. g indicates the number of ground states when $s=0$; when g is finite, it is equal to the number of low-temperature phases (the PS theory applies). g^* is the number of dominant ground states (equal to the number of low-temperature phases) of the stacked system (1.6).

We concentrate here and in Section 4.4 on an analysis of the low-temperature properties of the model for (α, h) in the (open) strip $\mathbb{P}$ with base BC. As discussed in ref. 4, stacking transforms the lines with nonzero entropy into lines on which the entropy is zero but the model is nonregular and the theory developed here does not apply. For other values of the parameters of Fig. 1, outside of the strip and the interval BC, the number of ground states is finite and the PS theory applies. On the interval DB the two ground states with plaquettes P^2_{nnn} are dominant and it is not hard to extend the arguments of ref. 4 to a proof of all the assumptions of Theorem A, yielding two pure low-temperature phases. We did not compute the complete phase diagram in the neighborhood of the interval DB. We give now the results concerning the strip; the details of the proof are in Section 4.5.

The ground states obtained by patching plaquettes of type P^1 can be described as follows. First, let G^1 denote the configuration equal to -1 at the points of $2 \cdot \mathbb{Z}^2$ and $+1$ otherwise. Let $\mathcal{G}^*$ denote the family of four configurations obtained from G^1 by applying translations. Then any ground state for (α, μ) in the strip can be obtained by first picking one of the configurations of $\mathcal{G}^*$, then picking one of the coordinate axes and a number of lines of the lattice perpendicular to the axis, and finally shifting the configurations on the chosen lines. It is clear that the family $\mathcal{G}$ of

configurations so obtained has zero entropy. Our assertion is that for small enough positive J the configurations of $\mathscr{G}^*$ yield upon stacking the dominant ground states of the model (1.6). We denote the family of these stacked configurations $\mathscr{G}^*$ again. As shown in Section 4.5, by Theorem A the model has exactly four low-temperature phases.

The domination of $\mathscr{G}^*$ is due to different kinds of excitations in different parts of the strip, and occurs in different orders. That is why the strip is partitioned into the subregions $\mathbb{P}^0$, $\mathbb{P}^2_{\text{nn}}$, and $\mathbb{P}^2_{\text{nnn}}$ on Fig. 1. For more details see Section 4.5.

2. DOMINATION, PEIERLS CONDITION, AND THE MAIN RESULTS

We consider here the ensembles of low-energy excitations around which we expand. These replace the ground states of PS theory (see BKL).

To motivate the following discussion of ground states and excitations, we now compare their properties in the PS theory and in the situations encountered in the present work.

Suppose, for simplicity, that the ground-state energy of the system is zero. For any configuration one can decompose the lattice into regions which contribute zero to the energy, and the rest, the "excited" regions. The zero-contributing region splits into connected components which are defined as the local ground states. The connected components of the rest form "excitations." In the PS theory, where the number of (global) ground states is finite, and therefore each of them is periodic, one can define things in such a way that each of the local ground states is a restriction of a global ground state. This implies, in particular, that any excitation without holes which is surrounded by a local ground state can be "removed" by extending the local ground state to a global one. This is not so in the cases considered in the present paper, when the number of global ground states is infinite, as is illustrated in Fig. 2. Here the removable excitation and the shadow of the large nonremovable excitation can be made as large as one may wish. All this necessitates a carefull treatment of local ground states, excitations, and free energies.

After introducing in Section 2.1 the notion of a removable excitation, we define the restricted ensembles of low-energy excitations (Section 2.2). These are dilute in the sense of BKL, i.e., they have a (standard) convergent low-temperature expansion. The locally dominant ground states are defined by the condition that their restricted ensembles have the lowest effective free energy. Using the low-temperature expansion, we then prove that when the number of the locally dominant ground states is finite, our

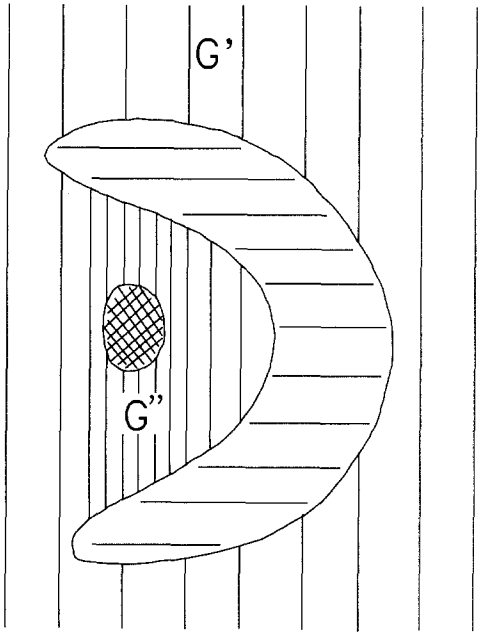

Fig. 2. The cross-hatched excitation is removable, the hatched one is not: there is no global ground state equal to G' and G'' on their respective domains.

version (2.9) of the Peierls condition is satisfied. Finally, in the last section, we formulate our main results and discuss a number of their variants and extensions.

2.1. Local Ground States and Excitations

If X is an element of $\mathscr{X}_\Lambda$ for some Λ, then X is a *partial configuration* and Λ is the *domain* of X, $\mathrm{dom}(X)$. For $M \subset \Lambda$ and $X \in \mathscr{X}_\Lambda$, X_M is the restriction of X to M.

Let l be a nonnegative number; for any subset Λ of the lattice its *l-boundary* is defined as[5]

$$\partial_l \Lambda = \{a \in \Lambda : \mathrm{dist}(a, \Lambda^c) < l\}$$

and we define the *l-neighborhood* $[\Lambda, l]$ and the *l-interior* $]\Lambda, l[$ of Λ as

$$[\Lambda, l] = \{a \in \mathbb{L} : \mathrm{dist}(a, \Lambda) \leqslant l\}, \qquad]\Lambda, l[= \{a \in \mathbb{L} : \mathrm{dist}(a, \Lambda^c) > l\}$$

[5] $\mathrm{dist}(a, b) = \max_i |a_i - b_i|$, $\mathrm{dist}(a, \Lambda) = \min_{b \in \Lambda} \mathrm{dist}(a, b)$.

By a *boundary* we understand an l-boundary with some fixed l. Later we will fix the value of l so that the retouch property, defined in the next section, holds.

Let Φ be a potential. For any finite B set

$$\phi_B = \min_X \Phi_B(X)$$

Φ is an *m-potential* if there exists a configuration X such that $\Phi_B(X) = \phi_B$ for any B.[32] To exhibit the structure of ground states more clearly, it is often convenient to reformulate the model in terms of an m-potential. We will introduce later the notion of an effective potential, which will play a similar role with respect to the low-temperature phases. Each of the models we consider has an m-potential. We emphasize that most systems admit m-potentials, even the so-called "frustrated" models. One can reformulate the theory in terms of the relative Hamiltonian, making it independent of the choice of the potential, as in ref. 29.

For an m-potential Φ let

$$\mathcal{G}_\Lambda = \mathcal{G}_\Lambda(\Phi) = \{X_\Lambda : X \in \mathcal{X},\ \Phi_B(X) = \phi_B \text{ if } B \subset \Lambda\}$$

Elements of $\bigcup \mathcal{G}_\Lambda$, where the sum is over all the finite subsets of the lattice, are called *local ground states*, and the elements of $\mathcal{G} = \mathcal{G}_{\mathbb{L}}$ are called (*global*) *ground states*. We stress that the local ground states are not restrictions of global ground states in general (see Fig. 2). $\mathcal{G}^*$ is the family of the dominant elements of $\mathcal{G}$, which are defined in Section 2.4 below. $\mathcal{G}^*$ is assumed to be finite; its elements are periodic but not necessarily equivalent under symmetries of the interaction.

A partial configuration X is an *excitation* if its restriction to the boundary of its domain is a (local) ground state. (Note that according to this definition any local ground state is an excitation, and that this notion is different from that of the excitation of ref. 32, though related to it.) A partial configuration X is said to be *contained* in a partial configuration Y, $X \subset Y$, if Y is equal to X on $\mathrm{dom}(X)$. An excitation which does not properly contain another one is an *elementary excitation*. Two excitations are *compatible* if there exists an excitation containing both of them. We comment in Remark 5 at the end of this section on the l (in)dependence of the construction.

Let H_0 be a Hamiltonian with an m-potential Φ_0, and let l be larger than the range of Φ_0; for an excitation X we define its *energy* as

$$H_0(X) = \sum_{B \subset \mathrm{dom}(X)} [\Phi_{0,B}(X) - \phi_{0,B}]$$

Defining the *support* of an excitation by

$$\mathrm{supp}(X) = \bigcup \{\mathrm{dom}(Y): \text{elementary excitation } Y \subset X\}$$

so that $\mathrm{supp}(X) = \mathrm{dom}(X)$ if X is elementary, we note that there exist $s_1, s_2 > 0$ such that for any excitation X

$$s_1 |\mathrm{supp}(X)| \leqslant H_0(X) \leqslant s_2 |\mathrm{supp}(X)|$$

which is also written as

$$s_1 |X| \leqslant H_0(X) \leqslant s_2 |X| \tag{2.1}$$

where $|X|$ stands for $|\mathrm{supp}(X)|$.

2.2. The Retouch Property

In the PS theory the ground states can be read off from their restrictions to any sufficiently large portion of the lattice, since their number is finite. This is no longer true when the number of ground states is infinite. However, in all our models *local ground states are determined by their restrictions to the boundaries*, i.e., two local ground states with a common domain which are equal on its boundary are equal everywhere. This uniqueness property is assummed in the following; it certainly does not hold in models with nonzero ground-state entropy.

An excitation X with a domain Λ is *removable* if it is equal to an element of $\mathcal{G}_\Lambda$ on the l-boundary of Λ; this element of $\mathcal{G}_\Lambda$, which is unique, since the ground states extend uniquely from the boundaries, is denoted $G(X)$. If G is a ground states and X is a removable excitation such that $G(X) = G$, then X will be called *an excitation of* G. If X is removable and $X \subset Y$, then there is a unique partial configuration *obtained by removing* X from Y, equal to $G(X)$ on $\mathrm{dom}(X)$ and to Y otherwise.

The models we consider, and many others, have the following *retouch property*: for any $E > 0$ there exists $l(E)$ such that if $l > l(E)$, then any elementary excitation with energy smaller than E is removable. (l is implicit in the definition of the boundary and therefore also in that of the excitation.) Note that by definition, in models with the retouch property the ground states extend uniquely from the boundaries. We will need the retouch property for excitations up to a certain energy, but in the models here it holds for any E.

The retouch property is akin to the decomposition property of ref. 20 (compare with the IM in one dimension, or the Pecherski model[32]). It will be proved to hold in a number of models in Section 4.

If X is any partial configuration equal to a ground state on the boundary of its domain, then there is a uniquely defined partial configuration *retouch of* X, $\mathrm{ret}(X)$, obtained from X by removing from it all elementary excitations with energy smaller than E. Here E and $l > l(E)$ are implicit.

The retouch property implies regularity of the model in the sense of ref. 31. We do not know whether the converse implication is true.

2.3. Restricted Ensembles

For a subset Λ of the lattice and a (possibly local) ground state G whose domain contains the l-neighborhood of Λ we define the *restricted ensembles*

$$\mathscr{X}^{G,E} = \{X \in \mathscr{X}_{\mathrm{dom}(G)} : X \text{ is an excitation and } \mathrm{ret}(X) = G\}$$
$$\mathscr{X}^{G,E}_{\Lambda} = \{X \in \mathscr{X}^{G,E} : X_{\mathrm{dom}(G)\setminus\Lambda} = G_{\mathrm{dom}(G)\setminus\Lambda}\}$$

For a Hamilton H with a potential Φ of a range $\leqslant l$ the corresponding partition function is

$$Z_E(\Lambda \mid G) = \sum_{X \in \mathscr{X}^{G,E}_{\Lambda}} \exp -\beta \sum_{B \cap \Lambda \neq \varnothing} \Phi_B(X)$$

We will need also partition functions defined by boundary conditions belonging to a restricted ensemble of a ground state. Thus, let Λ and G be as above and let $Y \in \mathscr{X}^{G,E}$. Then

$$Z_E(\Lambda \mid Y) = \sum_{X} \exp -\beta \sum_{B \cap \Lambda \neq \varnothing} \Phi_B(X)$$

where the summation is over $X \in \mathscr{X}^{G,E}_{\Lambda}$ such that the elementary excitations of X are compatible (in the sense of Section 2.1) with those elementary excitations of Y whose supports intersect the complement of Λ. The set of all such X is denoted $\mathscr{X}^{Y,E}_{\Lambda}$.

Let G be a ground state and let ϑ be a *multiplicity function* defined on the elementary excitations of G, i.e., a function from the set of elementary excitations of G to nonnegative integers which is nonzero on a finite set only. For a multiplicity function (m.f.) we set

$$H(\vartheta) = \sum_{X} \vartheta(X)\, H(X), \qquad \mathrm{supp}(\vartheta) = \bigcup_{\vartheta(X) \neq 0} \mathrm{supp}(X)$$

where

$$H(X) = \sum_{B \subset \mathrm{supp}(X)} [\Phi_B(X) - \Phi_B(G)]$$

$H_0(\vartheta)$ is the *energy* of ϑ and supp(ϑ) its *support*. For any excitation X we denote by ϑ_X the multiplicity function, equal to one on the elementary excitations of X and zero otherwise. Let

$$\phi_E(\vartheta, G) = \begin{cases} \exp - \beta H(\vartheta), & \text{if } \vartheta = \vartheta_X \text{ for some } X \in \mathscr{X}^{G,E} \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we define Boltzmann factors with boundary conditions: for a finite Λ, a ground state G defined on an l-neighborhood of Λ, and $Y \in \mathscr{X}^{G,E}$ we set

$$\phi_E(\vartheta, \Lambda, Y) = \begin{cases} \exp - \beta H(\vartheta), & \text{if } \vartheta = \vartheta_X \text{ for some } X \in \mathscr{X}_\Lambda^{Y,E} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$Z_E(\Lambda \mid Y) = \left\{ \exp \left[-\beta \sum_{B \cap \Lambda \neq \varnothing} \Phi_B(G) \right] \right\} \sum_{\vartheta} \phi_E(\vartheta, \Lambda, Y)$$

and the cluster expansion reads

$$\log Z_E(\Lambda \mid Y) = -\beta \sum_{B \cap \Lambda \neq \varnothing} \Phi_B(G) + \sum_{\vartheta} \phi_E^T(\vartheta, \Lambda, Y) \tag{2.2}$$

where the Ursell functions (truncated Boltzmann factors) $\phi_E^T(\vartheta, \Lambda, Y)$ are equal, up to a combinatorial factor, to $e^{-\beta H(\vartheta)}$ when ϑ is "connected" and zero otherwise. This follows from standard algebraic formalism and from the fact that $\phi_E(\vartheta, \Lambda, Y)$ can be written as $\phi_E^T(\vartheta, G)$ times a "one-point" factor equal to 0 or 1, depending on Y (see BKL, Appendix 1, for instance).

Terms of the cluster expansion are now classified according to their order of magnitude, as $\beta \to \infty$. We let

$$\mathscr{E} = \{H_0(X): \text{any excitation } X\}$$

be the set of all possible *energies of excitation*; $\mathscr{E} = \{0, E_1, E_2, \ldots\}$, with $0 < E_1 < E_2 = \ldots$, with $E_k \to \infty$ as $k \to \infty$. We note that $\mathscr{E}$ is an additive set of positive numbers, i.e., $E_i + E_j \in \mathscr{E}$ for any i, j. If $H_0(X) = E_k$, we say that X *is of order* k; similarly, if the energy $H_0(\vartheta)$ of an m.f. ϑ is E_k, we say that ϑ *is of order* k (such a k always exists and is unique).

In a by now standard way, one shows that there exists β_0 such that for $\beta > \beta_0$ the sum in (2.2) is absolutely convergent for each Λ and Y, uniformly in Y for each Λ. More precisely, one has the bound

$$\sum_{\vartheta: a \in \text{supp}(\vartheta), H_0(\vartheta) \geq E_k} |\phi_E^T(\vartheta, G)| \leq c_k e^{-\beta E_k} \tag{2.3a}$$

with c_k independent of G, β, and the lattice point a (see BKL, Appendix A), and for any a, b

$$\sum_{\vartheta: a, b \in \mathrm{supp}(\vartheta)} |\phi_E^T(\vartheta, G)| \leqslant c e^{-\beta\alpha|a-b|} \tag{2.3b}$$

with positive α and c, independent of a, b, G, and β. The bounds (2.3) hold when $\phi_E^T(\vartheta, G)$ is replaced by $\phi_E^T(\vartheta, \Lambda, Y)$, and then α and c are independent of Y and Λ, too.

If G is defined on all the lattice and is periodic, say $\hat{\mathbb{Z}}^d$ invariant, then

$$\log Z_E(\Lambda \mid Y) = -|\Lambda| f_E(G) + O(|\partial\Lambda|)$$

where

$$f_E(G) = \beta e_G - |\mathbb{L}/\hat{\mathbb{Z}}^d|^{-1} \sum_{\vartheta (\mathrm{mod}\, \hat{\mathbb{Z}}^d)} \phi_E^T(\vartheta, G), \qquad e_G = \lim_{\Lambda} |\Lambda|^{-1} \sum_{B \subset \Lambda} \Phi_B(G) \tag{2.4}$$

and the series is absolutely convergent (parameters on which H depends are implicit in f). We will need a localized version of these formulas in which the perturbative free energy of a ground state G is represented as a sum of local terms, with G neither periodic nor defined globally. More precisely, by grouping the terms of (2.2), we define an *effective potential* with the property that for the dominant ground states, and only for these, the effective potential is minimal. This is analogous to the concept of an m-potential. To allow for a flexibility of grouping which will be needed in our discussion of the models in Section 4, for each lattice point a we consider *weight functions* $B \mapsto \chi(a, B)$ and $\vartheta \mapsto \chi(a, \vartheta)$ defined on the sets of bonds and m.f.'s, respectively, such that $\chi \geqslant 0$ and for each B and ϑ

$$\sum_a \chi(a, B) = 1, \qquad \sum_a \chi(a, \vartheta) = 1$$

We then define

$$e_G(a) = \sum_B \chi(a, B)\, \Phi_B(G), \qquad f_E(G, a) = \beta e_G(a) + \tilde{f}_E(G, a)$$

where

$$\tilde{f}_E(G, a) = -\sum_{\vartheta} \phi_E^T(\vartheta, G)\, \chi(a, \vartheta)$$

and if G is a local ground state, then for points a which are far enough from the complement of $\mathrm{dom}(G)$ we define

$$f_{(E)}(G, a) = \beta e_G(a) - \sum_{H_0(\vartheta) \leqslant E} \phi_E^T(\vartheta, G)\, \chi(a, \vartheta) \tag{2.5}$$

"far enough" meaning that the m.f.'s appearing on the rhs here have support contained in dom(G). Finally, for any finite Λ and a ground state G defined on the l-neighborhood of Λ, we define

$$\tilde{f}_E(\Lambda, G, a) = -\sum_{\vartheta \subset \Lambda} \phi_E^T(\vartheta, G)\, \chi(a, \vartheta), \qquad f_E(\Lambda, G, a) = \beta e_G(a) + \tilde{f}_E(\Lambda, G, a)$$

The χ which we use below has the locality property that there exists N_1 such that

$$\begin{aligned} &\chi(a, B) = 0 \qquad \text{if } \operatorname{dist}(B, a) > N_1 \\ &\chi(a, \vartheta) = 0 \qquad \text{if } \operatorname{dist}(\operatorname{supp}(\vartheta), a) > N_1 \end{aligned} \tag{2.6}$$

Moreover, for ϑ of a high enough order we will set

$$\chi(a, \vartheta) = \frac{1}{|\operatorname{supp}(\vartheta)|} \qquad \text{if} \quad a \in \operatorname{supp}(\vartheta)$$

and $\chi(a, \vartheta) = 0$ otherwise; such χ will be called *standard*. Similarly, we call standard the weight function

$$\chi(a, B) = 1/|B| \qquad \text{if} \quad a \in B$$

and $\chi(a, B) = 0$ otherwise. Examples of weight functions can be found in Section 4. In a variation of this scheme employed in Sections 4.2 and 4.3, a will label figures of the lattice—elementary cubes or the M-figures of Section 4.2—and χ will be called standard (with respect to the family of figures) if $\chi(a, \vartheta)^{-1}$ [resp. $\chi(a, B)^{-1}$] is the number of figures containing the m.f. ϑ (resp. the bond B). While this definition of an effective potential is general enough to cover our examples, one can imagine a situation where a more compicated one may be needed.

Suppose now that (2.6) holds and that χ is standard for all ϑ of high enough order. Then (2.3) implies that there exists $\alpha > 0$ such that for any $E \geqslant E_k$ and only global ground state G

$$\begin{aligned} &\tilde{f}_E(G, a) - \tilde{f}_{(E_k)}(G, a) = O(e^{-\beta E_{k+1}}) \\ &\tilde{f}_E(\Lambda, G, a) - \tilde{f}_E(G, a) = O(e^{-\beta\alpha \cdot \operatorname{dist}(a, \Lambda^c)}) \end{aligned} \tag{2.7a}$$

and that, for G global or local, if $\operatorname{dist}(a, \Lambda^c) \geqslant N$, where N is a large enough constant independent of G and Λ, then

$$\tilde{f}_E(\Lambda, G, a) - \tilde{f}_{(E_k)}(G, a) = O(e^{-\beta E_{k+1}}) \tag{2.7b}$$

2.4. Domination and Peierls Condition

If the perturbative free energy of some of the ground states is smaller than that of the others, then it will be said that these ground states are *dominant*. More precisely, we will consider the following situation.

Let Φ_0 be a finite-range periodic potential for a Hamiltonian H_0 and let $\mathcal{G}^*$ be a finite family of (global periodic) ground states of H_0, $\mathcal{G}^* \in \mathcal{G}(\Phi_0)$. Furthermore, let $(H_i)_{i=1}^n$ be a finite family of Hamiltonians with finite-range periodic potentials $(\Phi_i)_{i=1}^n$, and for any parameters $\mu = (\mu_1, ..., \mu_n)$ we set

$$H_\mu = H_0 + \sum_i \mu_i H_i, \qquad \Phi_\mu = \Phi_0 + \sum_i \mu_i \Phi_i$$

so that Φ_μ is a potential for H_μ. We write sometimes Φ_B instead of $\Phi_{\mu,B}$.

We will assume given a curve $\mu(\cdot)$: $\beta \mapsto \mu(\beta)$ in the parameter space, β large enough, with

$$\mu(\beta) \to 0 \qquad \text{as} \quad \beta \to \infty \tag{2.8}$$

(usually exponentially fast; however, in the situation described at the end of this section, $\beta\mu(\beta)$ does not go to zero as $\beta \to \infty$). For the Hamiltonian $H_{\mu(\beta)}$ we will consider the reduced partition function with boundary conditions defined by an element Y of the restricted ensemble $\mathcal{X}^{G,E}$ of a global ground state G:

$$Z^0(\Lambda, \text{ret} = X \mid Y) = Z(\Lambda, \text{ret} = X \mid Y)/Z_E(\Lambda \mid Y)$$

Here the partition function $Z_E(\Lambda \mid Y)$ of the ensemble $\mathcal{X}_\Lambda^{Y,E}$ is as defined at the beginning of Section 2.3, X is a retouched excitation of G equal to G on the N-boundary of Λ (with N to be fixed later), and

$$Z(\Lambda, \text{ret} = X \mid Y) = \sum_{X' : \text{ret}\, X' = X} \exp -\beta \sum_{B \cap \Lambda \neq \varnothing} \Phi_B(X')$$

where the summation is over excitations X' of G equal to G outside of Λ with retouch equal to X, and such that the elementary excitations of X' are compatible (see the beginning of Section 2.3) with those elementary excitations of Y whose supports intersect the complement of Λ (see the beginning of Section 2.3). Whereas Z depends on the choice of the potential Φ_μ for H_μ, Z^0 does not.

Definition. With the above notation, we will say that the family $\mathcal{G}^*$ is dominant (in order D) on the curve $\mu(\cdot)$ if for any $E \geqslant E_D$ there

exists $N>0$ such that for any $G\in\mathscr{G}^*$ and Y, Λ, and X as above, the inequality

$$Z^0(\Lambda, \text{ret}=X|\,Y)\leqslant \exp\{-\beta' H_0(X)-c(E)\,e^{-\beta E_D}|\Lambda'(X)|\} \tag{2.9}$$

holds for β large enough, wtih $c(E)$ independent of $G\in\mathscr{G}^*$, Λ, and β, and $\beta'=O(\beta)$ as $\beta\to\infty$; here $\Lambda'(X)$ is the set of points of Λ at which X is equal to the nondominant ground states:

$$\Lambda'(X)=\Lambda\backslash(\text{supp}(X)\cup\{a\in\Lambda: X_{[a,N]}\in \text{pr}_{[a,N]}\mathscr{G}^*\})$$

We will call (2.9) *the fundamental estimation*; when it holds we will say that *the Peierls condition* is satisfied. In fact, it is enough to assume that (2.9) holds for $E=E_D$. Then the general case follows, possibly by going to large β.

To make the Peierls condition easier to check in a particular model, we introduce the following notion (this is especially helpful when perturbations of the original model are introduced and one considers the complete phase diagram).

Definition. The family $\mathscr{G}^*$ is locally dominant (in order D) on the curve $\mu(\cdot)$ if for any $E\geqslant E_D$ there exist effective potentials $f_{(E)}(\cdot,\cdot)$, and N, c, $\beta_0>0$ such that (i) if $G\in\mathscr{G}^*$, then $f_E(G, a)=f_E(G)$, and $f_E(G)$ is the same for all $G\in\mathscr{G}^*$, (ii) for any point a of the N-interior of dom(G)

$$f_{(E)}(G, a)\geqslant f_{(E)}(G^*, a) \tag{2.10a}$$

and if, in addition, $G_{[a,l]}\notin \text{pr}_{[a,l]}\mathscr{G}^*$, then

$$f_{(E)}(G, a)-f_{(E)}(G^*, a)\geqslant ce^{-\beta E_D} \tag{2.10b}$$

for $G^*\in\mathscr{G}^*$ and $\beta\geqslant\beta_0$. Here N has to be chosen in such a way that $f_{(E)}(G, a)$ is well defined, i.e., the m.f.'s which appear in it have support contained in dom(G). An effective potential satisfying (i) and (ii) will be called an *effective m-potential.*

Remarks. 1. Conditions (i) and (ii) can be replaced by conditions which are (seemingly) less stringent but which imply the existence of an effective potential satisfying (i) and (ii). First, we will usually check that (i) holds with $f_{(E)}$ in place of f_E. Then, by adjusting $\mu(\cdot)$ slightly [with a change of order $o(e^{-\beta E})$] one obtains a curve on which (i) holds. Second, in case the family H_μ removes the degeneracy of the ground states of $\mathscr{G}^*$, or removes the degeneracy up to a symmetry, it is enough to have that $f_{(E)}(G', a')-f_{(E)}(G'', a'')=o(e^{-\beta E_D})$ for any G', $G''\in\mathscr{G}^*$ and any lattice points a', a'', and that $a\mapsto f_{(E)}(G, a)$ has the symmetry of the ground state G. Then, again by adjusting $\mu(\cdot)$ slightly and by averaging the effective

potential over translations, one obtains an effective potential and a curve on which (i) holds. Finally, suppose that instead of (2.10b) one has that there exists a positive N' with the property that for any point a of the N-interior of dom(G) for which $G_{[a,l]} \notin \mathrm{pr}_{[a,l]}\mathcal{G}^*$ there is a lattice point b such that $|b-a| < N'$ and

$$f_{(E)}(G, b) - f_{(E)}(G^*, b) \geqslant ce^{-\beta E_D}$$

for $G^* \in \mathcal{G}^*$ and $\beta \geqslant \beta_0$, with N and N' so chosen that $f_{(E)}(G, b)$ is well defined. Then again taking averages of $f_{(E)}(G, b)$ over b in the N'-neighborhood of a one obtains an effective potential for which (2.10b) holds.

2. It is not hard to see that if the family $\mathcal{G}^*$ is locally dominant on $\mu(\cdot)$ and $\tilde{\mu}(\cdot)$ is another curve such that $\tilde{\mu}(\beta) - \mu(\beta) = o(e^{-\beta E_D})$, then $\mathcal{G}^*$ is locally dominant on $\tilde{\mu}(\cdot)$.

3. For most of the results it is sufficient to have $f_{(E)}$ defined in some finite range of E.

Proposition. If $\mathcal{G}^*$ is locally dominant, then $\mathcal{G}^*$ is dominant in the same order.

Proof. The proof depends on a rearrangement (2.16) of the low-temperature expansion of $\log Z^0$ which exhibits it in terms of differences of effective potentials for which the bounds (2.10) can be used. This yields the $|\Lambda'(X)|$ term in (2.9). Various error and boundary terms are absorbed into the difference between β' and β and into $c(E)$. The term $O(|\partial\Lambda|)$ does not appear, because of *exact* cancellations between the numerator and denominator of Z^0 at the boundary of Λ. A careful treatment of all these boundary and error terms makes the proof somewhat tedious.

The notation below is as in the definition of domination. The complement of supp(X) in Λ decomposes into a number of connected components, possibly only one; there are two components in Fig. 3, where supp(X) is shaded and the vertical and horizontal lines indicate $\Lambda'(X)$. Of these, one, the *outer* component, Λ_{ext}, extends to the boundary of Λ, while the others, Λ_i, have boundaries adjacent to supp(X); the outer component is bounded by supp(X) in part. Restriction of X to the l-neighborhood of Λ_i (resp. Λ_{ext}) defines a local ground state G_i (resp. G_{ext}). G_{ext} is equal to the dominant ground state $G = G(Y)$ on the N-boundary of Λ. We set $\Lambda_i' = \Lambda_i \cap \Lambda'(X)$. We consider first the quantity

$$\tilde{Z}^0 = \tilde{Z}(\Lambda, \mathrm{ret} = X \mid Y)/\tilde{Z}_E(\Lambda \mid Y) \tag{2.11}$$

where $\tilde{Z}_E(\Lambda \mid Y)$ differs from $Z_E(\Lambda \mid Y)$ in that the potential (Φ_B) is replaced by $[\Phi_B - \Phi_B(G)]$, whereas in the computation of $\tilde{Z}(\Lambda, \mathrm{ret} = X \mid Y)$ one is

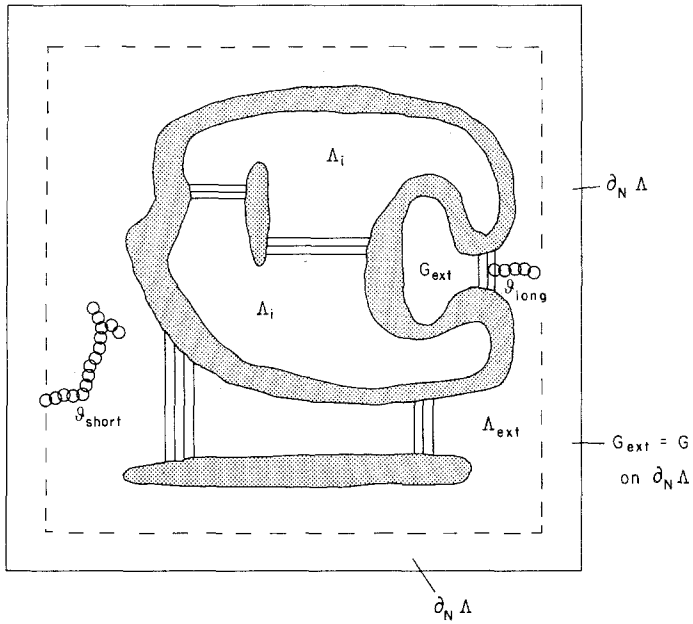

Fig. 3. Diagram for the proof of the fundamental estimation (2.9).

using the potential $\Phi_B - \Phi_B(G_i)$ for $B \subset \Lambda_i$, $\Phi_B - \Phi_B(G_{\text{ext}})$ for $B \subset \Lambda_{\text{ext}}$, and $\Phi_B - \Phi_B(G)$ for B intersecting $\text{supp}(X)$ (we could have chosen here $\Phi_B - \phi_{0,B}$ as well). We can write now

$$\tilde{Z}(\Lambda, \text{ret} = X | Y) = \{\exp[-\beta \tilde{H}(X)]\} \tilde{Z}_E(\Lambda_{\text{ext}} | Y, G_{\text{ext}}) \prod_i \tilde{Z}_E(\Lambda_i | G_i) \tag{2.12}$$

where

$$\tilde{H}(X) = \sum_{B: B \cap \text{supp}(X) \neq \varnothing} [\Phi_B - \Phi_B(G)]$$

$\tilde{Z}_E(\Lambda_i | G_i)$ is defined by the configurations of the restricted ensemble of G_i equal to G_i on the l-boundary of Λ_i^c, while in $\tilde{Z}_E(\Lambda_{\text{ext}} | Y, G_{\text{ext}})$ one sums over configurations Z of the restricted ensemble of G_{ext} equal to G_{ext} on the l-boundary of the complement of Λ_{ext}, and such that elementary excitations of Z are compatible with those elementary excitations of Y whose supports intersect the complement of Λ on the l-boundary of Λ. We note that because of (2.1) and the assumption (2.8), for β large enough

$$s_1' |X| \leqslant \tilde{H}(X) \leqslant s_2' |X|$$

for some positive s_1', s_2'. We now express the logarithms of the partition functions of the rhs of (2.12) in terms of the effective potential, and then do the same with $\log \tilde{Z}_E(\Lambda \mid Y)$, and finally we exhibit the cancellation in (2.9) which yield the fundamental estimation.

Substituting (2.12) into (2.11), we obtain

$$\log \tilde{Z}^0 = -\beta \tilde{H}(X) + \sum_i \log \tilde{Z}_E(\Lambda_i \mid G_i) + \log \tilde{Z}_E(\Lambda_{\text{ext}} \mid Y, G_{\text{ext}}) - \log \tilde{Z}_E(\Lambda \mid Y) \tag{2.13}$$

We show now that (2.13) can be rewritten as

$$\log \tilde{Z}^0 = -\beta_1 \tilde{H}(X) - \sum_i \sum_{a \in]\Lambda_i, N[} \tilde{f}_E(\Lambda_i, G_i, a) - \sum_{a \in \Lambda_{\text{ext}}, \operatorname{dist}(a, \operatorname{supp}(X)) \geqslant N} \tilde{f}_E(\Lambda_{\text{ext}}, G_{\text{ext}}, a) + \sum_{a \in \Lambda} \tilde{f}_E(\Lambda, G, a) + O(e^{-\beta \alpha N}) |\Lambda'(X)| \tag{2.14}$$

where $\beta_1 - \beta = O(e^{-\beta E_1})$, and α is as in (2.3) and (2.7).

To pass from (2.13) to (2.14), we first substitute for all the $\log \tilde{Z}_E$ their low-temperature expansion (2.2). For the interior regions we have

$$\log \tilde{Z}_E(\Lambda_i \mid G_i) = -\sum_{a \in]\Lambda_i, N[} \tilde{f}_E(\Lambda_i, G_i, a) + O(e^{-\beta E_1}) |\partial \Lambda_i| \tag{2.15}$$

which accounts for the first two terms on the lhs of (2.14). In particular, the boundary term in (2.15) is responsible in part for the change from β to β_1 $[\sum |\partial \Lambda_i| \leqslant O(|X|)]$.

Next we note that $\phi_E^T(\vartheta, Y)$ for ϑ at the boundary of Λ (ϑ_{short} on Fig. 3) is the same in $\log \tilde{Z}_E(\Lambda_{\text{ext}} \mid Y, G_{\text{ext}})$ and $\log \tilde{Z}_E(\Lambda \mid Y)$, except for the ϑ's intersecting both $\operatorname{supp}(X) \cup \Lambda'(X)$ and $\partial \Lambda$—the *long* ϑ's (ϑ_{long} on Fig. 3). The sum of $\phi_E^T(\vartheta, Y)$ over long ϑ has by (2.3b) a bound of the form $O(e^{-\beta \alpha N})[|\operatorname{supp}(X)| + |\Lambda'(X)|]$, since $\operatorname{dist}(\Lambda'(X) \cup \operatorname{supp}(X), \Lambda^c) \geqslant N$. Thus, by adding to both $\log \tilde{Z}_E(\Lambda_{\text{ext}} \mid Y, G_{\text{ext}})$ and $\log \tilde{Z}_E(\Lambda \mid Y)$ first *the same* combinations of $\phi_E^T(\vartheta, Y)$ with ϑ not long and then by adding combinations of $\phi_E^T(\vartheta, Y)$ with long ϑ we arrive at (2.14).

Relation (2.14) and the reasoning leading to it yield a similar expansion for Z^0:

$$\log Z^0(\Lambda, \text{ret} = X \mid Y) = -\beta_2 \tilde{H}(X) - \sum_i \sum_{a \in]\Lambda_i, N[} f_E(\Lambda_i, G_i, a) - \sum_{a \in \Lambda_{\text{ext}}, \operatorname{dist}(a, \operatorname{supp}(X)) \geqslant N} f_E(\Lambda_{\text{ext}}, G_{\text{ext}}, a) + \sum_{a \in \Lambda} f_E(\Lambda, G, a) + O(e^{-\beta \alpha N}) |\Lambda'(X)| \tag{2.16}$$

Indeed, we have

$$\log Z^0(\Lambda, \text{ret} = X \mid Y) = \log \tilde{Z}^0 + \beta \sum_i \sum_{B \subset \Lambda_i} [\Phi_B(G_i) - \Phi_B(G)] + \beta \sum_{B \subset \Lambda_{\text{ext}}} [\Phi_B(G_{\text{exp}}) - \Phi_B(G)] + o(\beta)\, O(|X|) \tag{2.17}$$

where the last term originates in B intersecting the support of X, and one has the factor $o(\beta)$ since, by assumption, $\Phi_{\mu(\beta),B} - \Phi_{0,B} \to 0$ as $\beta \to \infty$. To pass now from (2.14) to (2.16), we rearrange the sums of (2.17) into a sum of βe_G's, taking into account that linear combinations of terms of the form $\Phi_B(G_i)$ or $\Phi_B(G_{\text{ext}})$ which one may have to add in going from (2.14) to (2.16) are either canceled by the same combinations of $\Phi_B(G)$ (B at the boundary of Λ) or taken care off by changing slightly β_1.

To complete the proof of (2.9), we use (2.7) and (2.10) to get

$$\sum_{a \in]\Lambda_i, N[} [f_{(E)}(\Lambda_i, G_i, a) - f_{(E)}(\Lambda, G, a)] \geqslant O(e^{-\beta E_D}) |\Lambda_i'| \tag{2.18}$$

and

$$\sum_{a \in \Lambda_{\text{ext}}, \operatorname{dist}(a, \operatorname{supp}(X)) \geqslant N} [f_E(\Lambda_{\text{ext}}, G_{\text{ext}}, a) - f_E(\Lambda, G, a)] \geqslant O(e^{-\beta E_D}) |\Lambda_{\text{ext}} \cap \Lambda'(X)| \tag{2.19}$$

Using the bounds (2.18) and (2.19) for terms of (2.16), one finally obtains (2.9).

2.5. Main Results

As in the PS theory, we imbed the original Hamiltonian H_0 into a few-parameter family of Hamiltonians and analyze the phase diagram in the space of these parameters. We first state our main result, and variants of it, some of which are more suitable in the applications of Section 4. Let $H_1, \ldots, H_{g-1}$ be a family of finite-range Hamiltonians, and

$$H_\mu = H_0 + \sum_i \mu_i H_i$$

We say that we have a *complete phase diagram* in a subset U of the μ variables if there is an invertible map $\mu \mapsto t(\mu)$ such that both t and t^{-1} are Lipschitz continuous, and which maps U into a neighborhood of the origin

of the boundary of the positive octant $\{(x_G)\colon \min_G x_G = 0\}$ of $\mathbb{R}^g$ such that the pure thermodynamic phases (extremal periodic Gibbs states) are small perturbations of those ground states G for which $t(\mu)_G = 0$, as in the PS theory. Define the *perturbative phase diagram* by the map $\mu \mapsto t_E(\mu)$:

$$t_E(\mu)_G = f_E(\mu, G) - \min_{G'} f_E(\mu, G')$$

[f_E as in (2.5)], and the zero-temperature phase diagram by the map $\mu \mapsto t^0(\mu)$,

$$t^0(\mu)_G = e_G(H_\mu) - \min_{G'} e_{G'}(H_\mu)$$

where for a Hamiltonian H with a periodic potential Φ and a periodic configuration G we set

$$e_G(H) = \lim_{\Lambda} \frac{1}{|\Lambda|} \sum_{M \subset \Lambda} \Phi_M(G)$$

We say that *the set of perturbations removes the degeneracy of the ground states*, if the zero-temperature phase diagram is complete. This last condition is equivalent to the invertibility of the $(g-1)\times(g-1)$ matrix A with elements

$$A_{i,G} = e_G(H_i) - e_{G'}(H_i), \qquad i = 1, \ldots, g-1, \quad G, G' \in \mathcal{G}^*, \quad G \neq G'$$

Using (2.4) and standard bounds on ϕ^T, we obtain

$$\frac{d}{d\mu_i} f_E(\mu, G) = \frac{d}{d\mu_i} e_G(H_\mu) + O(e^{-\beta}) = e_G(H_i) + O(e^{-\beta}) \tag{2.20}$$

and also the bound

$$\left| \frac{d}{d\mu_i} f_E(\mu, G) \right| \leqslant \text{const} \tag{2.21}$$

From (2.20) and the inverse function theorem it follows that the perturbative phase diagram is complete, for μ small and β large, if the zero-temperature one is complete.

Theorem B. Let H_0 satisfy the retouch property and Condition ($\mathcal{L}$) of Section 3.3. Assume that we have a function $\beta \mapsto \mu_0(\beta)$, which tends to 0 as β goes to infinity, and a family $\mathcal{G}^*$ of g ground states of H_0 which is locally dominant in order D on the line $\mu_0(\cdot)$. Furthermore, assume that the family $H_1, \ldots, H_{g-1}$ removes the degeneracy of the ground states of $\mathcal{G}^*$.

Then for any large enough $E \geqslant E_D$ there exists $\beta_0(E)$ and $\alpha > 0$ such that for any $\beta > \beta_0(E)$ there is a complete phase diagram in the wedgelike region

$$|\mu - \mu_0(\beta)| \leqslant \alpha(\beta e^{\beta E_D})^{-3/2} \tag{2.22}$$

Moreover, this phase diagram is a small deformation of the perturbative one, in the sense that it is given by a map t such that

$$|t^{-1} - t_E^{-1}| \leqslant e^{-\beta E/2}$$

Extensions, Alternative Versions of Theorem B, and Comments

1. Taking E larger and larger, the last statement yields an asymptotic expansion of the phase diagram.

2. The 3/2 in (2.22) is replaced by another positive constant if the dimension is larger than 3.

3. One can prove a useful version of Theorem A by taking into account the symmetries of the system. Namely, let $\mathbf{G}$ be a group of symmetries of H_0, and suppose that each H_i, $i = 1,..., n$, is $\mathbf{G}$ invariant. Then one has a modification of Theorem A in which $\mathscr{G}^*$ is replaced by the family $\mathscr{G}^*/\mathbf{G}$ of classes of $\mathbf{G}$-equivalent ground states and the number n of perturbations is equal to the number of classes.

4. Assume that there is a set $\mathscr{K}$ in the space of the parameters (β, μ) and a finite family of ground states equivalent under symmetries of both H_0 and H_i which locally dominates to order E_D [with the constant c in (2.10) fixed throughout $\mathscr{K}$]. Then, in some neighborhood of $\mathscr{K}$, the pure phases are small perturbations of the ground states of $\mathscr{G}^*$.

Combining this remark with Theorem B, one can obtain by "patching" different regions of parameter space a complete description of the phase diagram in a domain larger than the one given by Theorem B. This is illustrated on some of the examples in Section 4.

5. The definition of the excitations and therefore also of the restricted ensembles depends on the constant l. However, using the cluster expansion, one can see that with changed l one obtains the same $f_{(E)}$, and therefore the same perturbative phase diagram.

6. Instead of assuming that the perturbations remove the degeneracy of the ground states, we can consider the more general situation where the perturbations remove the degeneracy of the restricted ensembles, i.e., they change the energy of the excitations, but possibly not of the ground states. A simple example where this occurs is given by the Blume–Capel model

(for a detailed discussion of this model from the point of view of the PS theory, see ref. 32). This is a spin-1 model, with spin variables s_a taking values $-1, 0, +1$ and with the Hamiltonian

$$H_0 = \sum_{\text{n.n.}} (s_a - s_b)^2$$

where the sum extends over all pairs of nearest neighbors. As explained in ref. 32, there are three ground states G^+, G^0, G^- of H_0, equal $+1, 0, -1$everywhere, respectively, and the ground state G^0 is dominant. Now, we consider a one-parameter family of perturbations $\mu \sum_a \phi_a$, where $\phi_a(X) = -1$ if $X_a = 0$, $X_{a-e_1} = X_{a+e_1} = +1$ and $\phi_a(X) = 0$ otherwise.

Clearly, this perturbation does not change the ground states (for μ small), but it does affect the energies of the excitations. Actually, when $\mu > 0$, the G^+ ground state is favored, and there exists a line

$$\mu(\beta) = \frac{1}{\beta} \ln 2 + O(e^{-c\beta}) \tag{2.23}$$

on which the G^0 and the G^+ phases coexist. Note that $\beta\mu(\beta)$ does not approach 0 as $\beta \to \infty$ any more. The calculations which yield (2.23) are straightforward; in more complicated cases the Newton-polygon techniques should prove useful.

With this kind of system in mind, we say that a family $H_1, \ldots, H_{g-1}$ of perturbations *removes the degeneracy of the restricted ensembles* if the matrix A,

$$A_{i,G} = \langle H_i \rangle_E^G - \langle H_i \rangle_E^{G'}, \qquad G \neq G', \quad i = 1, \ldots, g-1$$

is invertible, where $\langle H \rangle_E^G$ denotes the mean value of the energy in the restricted ensemble of the excitations of the ground state G with energy less than E (thus, A depends on E).

In that case, there exists a c such that $\|A^{-1}\| \leqslant e^{c\beta}$ (because all the matrix elements of A, and therefore also the determinant of A, have a convergent low-temperature expansion). Assume now that we have a line $\mu_0(\beta)$ on which the ground states of $\mathcal{G}^*$ dominate in order E_D, and the free energies $f_E(G)$ are independent of $G \in \mathcal{G}^*$. Assume that the family $H_1, \ldots, H_{g-1}$ of perturbations removes the degeneracy of the restricted ensembles in the above sense. Then, if $E = E(E_D, c)$ and β are large enough, there is a complete phase diagram in the wedgelike domain:

$$|\mu - \mu_0(\beta)| \leqslant e^{-c'\beta}$$

with $c' > \max(c, \frac{3}{2}E_D)$. One can also prove a version of this in the presence of symmetries.

3. CONTOUR MODELS, PEIERLS BOUND, AND PROOFS

We introduce here first the large-scale contours, and then write down the recursion realtions satisfied by the various partition functions (Section 3.2), prove the Peierls bound (Section 3.3), introduce the contour models (Section 3.4), and finally in Section 3.5 we prove the theorems.

3.1. Contours

Let L be the smallest even integer $\geqslant (\beta e^{\beta E_D})^{1/2}$ and let C_0 be the (closed) cube centered at zero with an edge length L. C_0 will also stand for the intersection of the cube with the lattice. For any $a \in \mathbb{Z}^d$

$$C_a = C_0 + \tfrac{1}{2}La$$

will be called a *cube* in this section; the "cubes" of the next section are smaller. We note that if two cubes, say C and D, do not overlap, then $\mathrm{dist}(C, D) \geqslant L/2$, where the distance is defined as in Section 2.1. A *region* is a subset of the lattice which is a union of a finite collection of cubes or a finite connected component of the complement of such a family. $|\Lambda|_L$ will stand for the number of cubes contained in Λ, and for any collection M of cubes, $\underline{M}$ will denote the union of cubes of M.

Fix $l < L$, and let G be a ground state defined in the l-neighborhood $[C, l]$ of C; C is a G-cube of a configuration X if X is defined on $[C, l]$ and $\mathrm{ret}(X)_{[C,l]} = G_{[C,l]}$; C is a *regular cube* of X if it is a G-cube of X for some $G \in \mathcal{G}^*$. Otherwise, C is *an irregular cube* of X.

Irregular cubes can be two types. Cubes of *type I* are G-cubes for some $G \in \mathcal{G} \backslash \mathcal{G}^*$. Hence, if C is a (*type II*) *cube* of a configuration X, then there is an elementary excitation of X of large energy with support intersecting C. As in the PS theory, one has an *extension property* of the ground states of $\mathcal{G}^*$: regular overlapping cubes are of the same kind, i.e., they correspond to the same elements of $\mathcal{G}^*$. Connected families of cubes and components of such families are defined in the usual way.

A *contour* Γ is a pair (M, X), where M is a finite component of the family of the irregular cubes of a configuration, say Y, and X is the restriction to the l-neighborhood of $\underline{M}$ of the configuration obtained from Y by removing all the low-energy excitations with support not intersecting $\underline{M}$. We need this somewhat involved definition to avoid troubles with excitations whose supports intersect the l-neighborhood of $\underline{M}$. We call M the *support* of Γ, $\mathrm{supp}(\Gamma)$.

We adapt to our situation many definitions of the PS theory. In

particular, for a contour Γ and G in $\mathscr{G}^*$ we introduce the notion of its *G-interior*, $\mathrm{Int}_G(\Gamma)$, of *G-contours*, and we set

$$\mathrm{Int}(\Gamma) = \bigcup_{G \in \mathscr{G}^*} \mathrm{Int}_G(\Gamma)$$

and

$$v_G(\Gamma) = |\mathrm{Int}_G(\Gamma)|, \qquad v(\Gamma) = |\mathrm{Int}(\Gamma)|$$
$$|\Gamma| = |\mathrm{supp}(\Gamma)|, \qquad |\Gamma|_L = |\mathrm{supp}(\Gamma)|_L$$

$|M|_L$ denotes the number of cubes in M; $|M|_L$ is of the same order as the number of cubes contained in M (in an obvious sense). We write $M \subset \Lambda$ if each cube of M is contained in Λ. For any family ω of contours, $\theta(\omega)$ is the union of supports of the contours of ω and their interiors; we write $\theta(\Gamma)$ instead of $\theta(\{\Gamma\})$. One defines also the *contour map* γ (resp.: *exterior contour map* γ^{ext}), which assigns to every configuration the family of all its contours (resp.: its exterior contours).

3.2. Recursion Relations

For any $G \in \mathscr{G}^*$, $Y \in \mathscr{X}^{G,E}$, and a region Λ we set

$$Z^0(\Lambda \mid Y) = Z(\Lambda \mid Y)/Z_E(\Lambda \mid Y)$$

Here

$$Z(\Lambda \mid Y) = \sum_X \exp -\beta \sum_{B \cap \Lambda \neq \varnothing} \Phi_B(X)$$

where the summation is over (global) configurations X equal to G on the complement of Λ and such that (1) the elementary excitations of X are compatible with those elementary excitations of Y whose supports are not contained in Λ, and (2) each irregular cube of X is contained in $]\Lambda, L/2[$. The $Z_E(\Lambda \mid Y)$ is defined by restricting the summation to X with retouch equal to G (see the beginning of Section 2.3). One has

$$Z^0(\Lambda \mid Y) = \sum_\omega Z^0(\Lambda, \gamma^{\mathrm{ext}} = \omega \mid Y) \tag{3.1}$$

where the sum is over families of mutually exterior G-contours with support contained in $]\Lambda, L/2[$. For any such family ω define its *interaction energy* $U_\Lambda(\omega \mid Y)$ by

$$Z^0(\Lambda, \gamma^{\mathrm{ext}} = \omega \mid Y) = e^{-U_\Lambda(\omega \mid Y)} \prod_{\Gamma \in \omega} Z^0(\Lambda, \gamma^{\mathrm{ext}} = \{\Gamma\} \mid Y) \tag{3.2}$$

U_Λ has its origin in fluctuations in the restricted ensemble and it can expressed through the corresponding partition functions:

$$e^{-U_\Lambda(\omega \mid Y)} = \frac{Z_E(\Lambda \backslash \theta(\omega) \mid Y, \omega)/Z_E(\Lambda \mid Y)}{\prod_{\Gamma \in \omega} [Z_E(\Lambda \backslash \theta(\Gamma) \mid Y, \Gamma)/Z_E(\Lambda \mid Y)]} \tag{3.3}$$

where $Z_E(\cdot \mid Y, \omega)$ has boundary conditions defined in part by Y and in part by the contours of ω, in the obvious sense; $Z_E(\cdot \mid Y, \Gamma)$ and $Z^0(\cdot \mid \Gamma)$ below are defined similarly. Furthermore, define the surface part $\psi_\Lambda(\Gamma \mid Y)$ of the contour free energy through

$$\begin{aligned} Z^0(\Lambda, \gamma^{\text{ext}} = \{\Gamma\} \mid Y) &= \exp[-\psi_\Lambda(\Gamma \mid Y)] \prod_{G \in \mathcal{G}^*} \exp\{v_G(\Gamma)[f_E(\mu \mid G(Y)) - f_E(\mu \mid G)]\} \\ &\quad \times Z^0(\text{Int}_G(\Gamma) \mid \Gamma) \end{aligned} \tag{3.4}$$

By the obvious identity

$$Z^0(\Lambda, \gamma^{\text{ext}} = \{\Gamma\} \mid Y) = Z^0(\Lambda, \gamma = \{\Gamma\} \mid Y) \prod_{G \in \mathcal{G}^*} Z^0(\text{Int}_G(\Gamma) \mid \Gamma)$$

this definition is equivalent to

$$\psi_\Lambda(\Gamma \mid Y) = -\log Z^0(\Lambda, \gamma = \{\Gamma\} \mid Y) + \sum_{G \in \mathcal{G}^*} v_G(\Gamma)[f_E(\mu \mid G(Y)) - f_E(\mu \mid G)] \tag{3.5}$$

When all elements of $\mathcal{G}^*$ are equivalent and $\mu = 0$, or, more generally, when $\mu = \mu_0$ and the condition (i) of the definition of local domination is satisfied, (3.4) takes the form

$$Z^0(\Lambda, \gamma = \{\Gamma\} \mid Y) = e^{-\psi_\Lambda(\Gamma \mid Y)} \tag{3.6}$$

The infinite-volume limits

$$\psi(\Gamma) = \lim_\Lambda \psi_\Lambda(\Gamma \mid Y), \qquad Z(\Gamma) = \lim_\Lambda Z^0(\Lambda, \gamma^{\text{ext}} = \{\Gamma\} \mid Y)$$

exist and depend on Y through $G(Y)$ $[= G(\Gamma)]$ only; $Z(\Gamma)$ corresponds to the crystal partition function of PS. With these definitions, taking the infinite-volume limit in (3.4), one obtains the identity

$$Z(\Gamma) = e^{-\psi(\Gamma)} \prod_{G \in \mathcal{G}^*} \exp\{v_G(\Gamma)[f(\mu \mid G(\Gamma)) - f(\mu \mid G)]\}\, Z^0(\text{Int}_G(\Gamma) \mid \Gamma) \tag{3.7}$$

Defining

$$W_\Lambda(\omega \mid Y) = U_\Lambda(\omega \mid Y) - \sum_{\Gamma \in \omega} [\psi_\Lambda(\Gamma \mid Y) - \psi(\Gamma)] \tag{3.8}$$

the *fundamental recursion relations* take the form of a combination of (3.7) with

$$Z^0(\Lambda \mid Y) = \sum_{\omega} e^{-W_\Lambda(\omega \mid Y)} \prod_{\Gamma \in \omega} Z(\Gamma) \tag{3.9}$$

where the summation is over all the families ω of exterior contours, as in (3.1). μ is here implicit, and $Z(\Gamma)$ can be expressed through the reduced partition functions, as in (3.7).

The interdependence between the contours is expressed by W, which can be viewed as a weak many-body interaction between them. We shall show that for large β this interaction is weak indeed.

3.3. The Peierls Bound

We prove here our main estimate, (3.10) [or (3.11)] below—*the Peierls bound.* It says that the activities of the large-scale contours of Section 3.1 are small, which will imply that in the Gibbs states defined by the dominant ground states, the contours are unlikely. This estimate is similar to that of the hypothesis A2i of BKL. It could be called a Peierls condition for (large-scale) contours. It is proved below to hold under the following additional hypothesis.

Condition ($\mathscr{L}$). If C is a cube of edge length L and $G \in \mathscr{G}_C \backslash \mathscr{G}_C^*$, then $|\Lambda'(G)| \geqslant L^2$.

In the models considered later, Condition ($\mathscr{L}$) is satisfied, since for each cube of type II, $\Lambda'(G)$ consists of a number of planes, or layers. [We expect Condition ($\mathscr{L}$) to be in general related to the degree of degeneracy of ground states, as discussed in Section 5.5.]

Lemma. The retouch property and Condition ($\mathscr{L}$) imply that if the ground states of $\mathscr{G}^*$ are dominant on a curve $\mu_0(\cdot)$ and the cutoff energy E of the effective potential is chosen large enough (for instance, larger than $\frac{3}{2}E_D$ in three dimensions), then there exists $c_1 > 0$ such that for all large enough β, for any region Λ of the lattice and any collection M of cubes contained in the $L/2$-interior of Λ

$$\sum_{\mathrm{supp}(\Gamma) = M} e^{-\psi_\Lambda(\Gamma \mid Y)} \leqslant e^{-\rho |M|_L} \tag{3.10}$$

where $\rho = c_1\beta$, ψ_1 is given by (3.5) with $\mu = \mu_0(\beta)$, and Y is any configuration of the restricted ensemble of a ground state of $\mathscr{G}^*$. (We recall that the contours are defined on a temperature-dependent scale.)

(A simple case, which, however, illustrates the most important points of the proof, can be found in the Appendix to ref. 4; an error in that appendix is corrected below.) We have to prove that

$$\sum_{\text{supp}(\Gamma) = M} Z^0(\Lambda, \gamma = \{\Gamma\} \mid Y) \leqslant e^{-\rho |M|_L} \tag{3.11}$$

[see (3.6)]. To deduce (3.11) from the Peierls condition (2.5), we estimate the lhs of (3.11) by first fixing a subfamily M^2 of M and restricting the summation to those contours Γ which have M^2 as the family of cubes of type II. We obtain

$$\sum_{X:\, \text{ret}, M^2(X) = M^2} \sum_{\text{ret}(\Gamma) = X} Z^0(\Lambda, \gamma = \{\Gamma\} \mid Y) \tag{3.12}$$

where the first sum is over the family of all the retouched excitations X having M^2 as the family $M^2(X)$ of cubes of type II, and the second is over contours $\Gamma = (M, X')$ for which $\text{ret}(X') = X$. The last condition is written as $\text{ret}(\Gamma) = X$. By $M^1(X)$ we denote the family of cubes of type I of X (Section 3.1).

Using the fundamental estimation (2.5), we bound (3.12) by

$$\sum_{X:\text{ret}, M^2(X) = M^2} \exp\{-\beta' H(X) - c[\exp(-\beta E_D)]\, |\Lambda'(X)|\} \tag{3.13}$$

By definition of $M^2(X)$ for each cube of M^2 there is a large-energy elementary excitation of X with support intersecting the cube; by the Condition $(\mathscr{L})$, $|\Lambda'(X)| \geqslant \alpha L^2 |M^1|_L$ (where α^{-1} is the number of cubes overlapping with any fixed cube). Therefore (3.13) is bounded by

$$\sum_{M^1 \cup M^2 = M,\, M^1 \cap M^2 = \varnothing} \exp[-\alpha L^2 |M^1|_L \exp(-\beta E_D)] \times \sum_{X:\text{ret}, M^2(X) = M^2} \exp[-\beta' H(X)] \tag{3.14}$$

Clearly, if we show that

$$\sum_{X:\text{ret}, M^2(X) = M^2} e^{-\beta' H(X)} \leqslant (cL^3 e^{-\beta' E})^{\alpha' |M^2|_L} \tag{3.15}$$

with $\alpha' > 0$, we can finish the proof of the Lemma as follows: Insert (3.15) into (3.14) and use our definition of L $[= L(\beta) = (c\beta e^{\beta E_D})^{1/2}]$, and choose

$E \geqslant 3E_D$, which imply that $L^2 e^{-\beta E_D} = c\beta$ while $L^3 e^{-\beta' E} \leqslant e^{-c'\beta E}$ for β large. Then, since $|M|_L = |M^1|_L + |M^2|_L$ and since the first sum in (3.4) contains at most $2^{|M|_L}$ terms, (3.10) follows. We now prove (3.15).

Observe first that for a fixed cube C

$$\sum_{X:\text{ret}, M^2(X)=C} e^{-\beta' H(X)} \leqslant cL^3 e^{-\beta' E} \tag{3.16}$$

Indeed, let $X_1, \ldots, X_n$ be the elementary excitations of X. The $\text{supp}(X_i)$ is connected for each i, $\text{supp}(X_i) \cap C \neq \varnothing$ and $H(X_i) \geqslant E$ (X is retouched). Therefore the lhs of (3.16) is bounded by

$$\sum_{n=1}^{\infty} \left(\sum_{X:\text{supp}(X)\cap C\neq\varnothing, \text{supp}(X)\text{conn}, H(X)\geqslant E} e^{-\beta H(X)} \right)^n \tag{3.17}$$

The sum starts with $n=1$ because of the condition $M^2(X) = C$. Clearly, by a standard Peierls estimate,

$$\sum_{X:\text{supp}(X)\cap C\neq\varnothing, \text{supp}(X)\text{conn}, H(X)\geqslant E} e^{-\beta H(X)} \leqslant cL^3 e^{-\beta E} \tag{3.18}$$

$L^3 = |C|$ counts the number of possible "origins" of the connected set $\text{supp}(X)$. Inserting (3.18) into (3.17) yields (3.16), since $L^3 e^{-\beta E}$ is small, and $e^y - 1 \approx y$ for y small.

We now return to the proof of (3.15). We would like to use (3.16) and to extract from (3.15) a factor equal to the rhs of (3.16) for each C in M^2. We cannot quite do that because the same connected components of X may intersect several cubes and we would have overcounting. [This was the error in (A.23) of ref. 4.] To avoid that, let $\tilde{M}^2$ be a maximal subfamily of M^2 with the property that it does not contain any pair of cubes with distance less than L (in units of the original lattice). Then if C, C' belong to $\tilde{M}^2$, $[C, L/4]$ and $[C', L/4]$ do not intersect.

Let $X_1, \ldots, X_n$ be the connected components of X. Observe [use (2.1)] that since $\text{supp}(X)$ is connected and $H(X_i) \geqslant E$,

$$H(X_i \cap [C, L/4]) \geqslant E \tag{3.19}$$

if $\text{supp}(X) \cap C \neq \varnothing$ (since L is large).

Let $X_1, \ldots, X_k$, $k \leqslant n$, be the connected components of X intersecting a cube of $\tilde{M}^2$. Then

$$\sum_{(X_i)_{i=k+1}^n} \prod_i \exp[-\beta' H(X_i)] \leqslant \exp\{c[\exp(-\beta' E)]\,|\cup M^2|\} \tag{3.20}$$

by an estimate similar to (3.16), except that the sum [see (3.17)] does not necessarily start with 1 (there may not be any X_i, $i > k$). We have

$$(3.20) \leqslant \exp[\exp(-c\beta')]\,|M^2|_L$$

for some $c > 0$, because $|M^2| \approx L^3 |M^2|_L$ and

$$L^3 e^{-\beta' E} \leqslant e^{-c\beta}$$

We fix now the restrictions of X_i to $[C, L/4]$, $i = 1,..., k$ so that

$$H(X_i) = \sum_{C \in \tilde{M}^2} H(X_i \cap [C, L/4]) + H(\text{rest})$$

and sum over the remaining parts of the X_i. This sum is bounded by

$$\exp\left\{[\exp(-c\beta)] \sum_{i=1}^{k} \sum_{C \in \tilde{M}^2} |X_i \cap [C, L/4]|\right\} \tag{3.21}$$

for some $c > 0$; this is similar to (3.20). We get $e^{-c\beta}$ instead of $e^{-\beta E}$ in (3.18) because the remaining part need not have energy $\geqslant E$. However, since the X_i are connected, these remaining parts must be connected to some $X_i \cap [C, L/4]$ (for some i or C), which gives the volume term in (3.21).

Finally, we sum over $X_i \cap [C, L/4]$. These are disconnected and (3.21) gives only a small correction to $\beta' H(X_i \cap [C, L/4])$. Using (3.19), we get a bound like (3.15) from this sum for each cube in $\tilde{M}^2$. But $|\tilde{M}^2|_L \geqslant \alpha' |M^2|_L$ for some $\alpha' > 0$, which yields (3.15).

3.4. Contour Models

The main idea in the PS theory is to relate the real models to contour models. Here, following BKL, we define our version of the contour models. However, unlike in the PS theory, where the contours interact only through hard-core exclusion, we have to consider so-called "interacting contour models" where on top of the hard-core exclusion one has a weak long-range many-body interaction, given here by $W_A(\omega, Y)$ in (3.8). This generalization is encountered, in one form or another, in most extensions of the PS theory.[3,9,44] We regard it as more or less standard. Therefore, we shall simply recall here the definitions and results, refering to BKL, which we shall closely follow, for the details of the proofs.

We define a *contour functional* F as a real-valued function on the set of contour configurations satisfying

$$\|F\| = \sup_{\Gamma} [|F(\Gamma)|/|\underline{\text{supp}(\Gamma)}| < \infty \tag{3.22}$$

and a τ-functional as a contour functional for which

$$\sum_{\mathrm{supp}(\Gamma)=M} e^{-F(\Gamma)} \leqslant e^{-\tau|M|_L} \tag{3.23}$$

This is the analogue of BKL, (3.12); (3.13) of BKL is not needed. As in BKL, we observe that ψ_Λ are τ-functionals, by (3.10). Given a τ-functional F, we define the *dilute* partition function

$$Z(\Lambda; F|Y) = Z(\Lambda; F, \mu|Y) = \sum_{\omega} e^{-W_\Lambda(\omega|Y)} \prod_{\Gamma\in\omega} Z(\Gamma; F) \tag{3.24}$$

where W is defined by (3.8), μ is implicit in W, and the summation extends over all families ω of mutually exterior $G(Y)$ contours, as in (3.1); and the *crystal* partition function

$$Z(\Gamma; F) = e^{-F(\Gamma)} \prod_{G\in\mathscr{G}^*} Z(\mathrm{Int}_G \Gamma; F|\Gamma) \tag{3.25}$$

where, as in Section 3.2, $Z(\mathrm{Int}_G \Gamma; F|Y)$ with any Y having Γ as a contour. These partition functions are defined recursively by the identities (3.24) and (3.25), starting with $Z(\Lambda; F|Y) = 1$ if no $G(Y)$ contour is contained in Λ. They depend on μ via W. Note that the relation betwen crystal and dilute partition functions is similar to the fundamental recursion relation (3.9).

One has the following estimates on the free energy of the contour models: Let $Y\in\mathscr{X}^{G,E}$,

$$s(F, \mu) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log Z(\Lambda; F|Y)$$

The limit here exists, and is independent of the boundary condition $Y\in\mathscr{X}^{G,E}$. We write

$$\log Z(\Lambda; F|Y) = s(F, \mu)|\Lambda| + A(\Lambda; F, \mu|Y) \tag{3.26}$$

Note that we define $s(F, \mu)$ on the scale of the lattice $\mathbb{L}$, while it would be natural to define it on the large scale (i.e., by dividing by $|\Lambda|_L$ instead of $|\Lambda|$), since the contours are defined on the scale. We do it because it is more convenient to have all the free energies defined on the same scale. Note, however, that with this definition we have the bounds

$$|s(F, \mu)| \leqslant o(e^{-\tau})L^{-3}, \qquad |A(\Lambda; F, \mu|Y)| \leqslant o(e^{-\tau})\,|\partial\Lambda|\,L^{-3} \tag{3.27}$$

The proof of (3.27) is fairly standard (see BKL for details). Inserting (3.24) into (3.25) and iterating, one obtains a sum over all families of contours,

not necessarily exterior ones [see BKL, (A.17)]. These contours interact via hard-core exclusion and via W, denoted $\bar{W}$ in BKL. Inserting the cluster expansion (2.2) into the definitions (3.3) of U and (3.8) of W, one sees, as in BKL, (A.19), that all $\phi_E^T(\vartheta, Y)$ cancel out except possibly those for which $\sup(\vartheta)$ intersects the support of several contours or the suport of one contour and the boundary of Λ (these latter terms are absent in BKL, Appendix 2). In order to work always on the L lattice, let us define the interactions on the L lattice by resumming all ϑ's contained in a given set of cubes. Now, we combine the following three facts: (1) the distance between two contours, in units of the original lattice $\mathbb{L}$, is at least L, and similarly for the distance between a contour and the boundary of Λ; (2) standard estimates show that $\phi_E^T(\vartheta | Y)$ decay exponentially with $\operatorname{supp}(\vartheta)$ [BKL, (A.21)]; (3) $\phi_E^T(\vartheta | Y) = 0$ unless ϑ is connected. Then we obtain that the interaction between the contours is very small indeed: it decays like e^{-Ld}, where d is the distance in units of L. This still holds after the resummation mentioned above, because the latter multiplies this bound by a factor of order L^3 at most. Also, in the bounds on the derivatives with respect to μ (BKL, pp. 535–536), one may replace $O(1)$ by $e^{-O(L)}$. We shall not discuss these improvements in more detail, since we shall not need their full strength. Then we can perform a high-temperature expansion of the interactions in W, as in BKL, (A.22), obtain a polymer model like BKL, (A.25), where the polymers are made of cubes, and derive (3.27) from standard estimates on polymer models. The factor L^{-3} in (3.27) comes simply from bounds similar to BKL, (3.15) and (3.16), if one defines the free energy per unit of the large-scale lattice. We also have standard Lipschitz estimates:

$$|s(F_1, \mu_1) - s(F_2, \mu_2)| \leqslant O(e^{-\tau})(\|F_1 - F_2\| + |\mu_1 - \mu_2|)$$

and

$$|\Delta(\Lambda; F_1, \mu_1 | Y) - \Delta(\Lambda; F_2, \mu_2 | Y)| \leqslant O(e^{-\tau})(\|F_1 - F_2\| + |\mu_1 - \mu_2|) \quad (3.28)$$

We shall also need contour models with parameter. These are defined by the partition function

$$Z(\Lambda; F, b, \mu | Y) = \sum_{\omega} e^{bv(\omega)} e^{-W_\Lambda(\omega | Y)} \prod_{\Gamma \in \omega} Z(\Gamma; F) \quad (3.29)$$

and satisfy the following estimates: Define Δ by the relation

$$\log Z(\Lambda; F, b, \mu | Y) = [s(F) + b] |\Lambda| + \Delta(\Lambda; F, b, \mu | Y)$$

Then

$$\Delta(\Lambda; F, b, \mu | Y) \leqslant O(e^{-\tau}) |\partial \Lambda|$$

and

$$\begin{aligned}|\Delta(\Lambda; F_1, b_1, \mu_1 | Y_1) - \Delta(\Lambda; F_2, b_2, \mu_2 | Y_2)| \\ \leqslant 2|b_1 - b_2|\,|\Lambda| + 2|\Lambda| e^{-\tau(1-\delta(\Lambda)/2}\,|||F_1 - F_2||| \\ + O(1)|\mu_1 - \mu_2| + O(e^{-\beta})\,|\partial\Lambda|\end{aligned} \tag{3.30}$$

with

$$|||F||| = \sup_{\Gamma} \frac{|F(\Gamma)|}{|\theta(\Gamma)|} e^{-(\tau/2)\delta(\Gamma)} \tag{3.31}$$

where $\delta(F)$ is the diameter in the large-scale units, while $|\theta(\Gamma)|$ is the small-scale size of $\theta(\Gamma)$. With this norm one has also

$$|s(F_1, \mu_1) - s(F_2, \mu_2)| \leqslant O(e^{-\tau})(|||F_1 - F_2||| + |\mu_1 - \mu_2|) \tag{3.32}$$

The bounds (3.30) [resp. (3.32)] are analogous to BKL, (3.21) [resp. BKL, (3.17)], and the proof is as in BKL, except that for (3.32) one uses the extra factor of $|C| = L^3$ in the definition of $s(F, \mu)$ in order to cancel out the one coming from $|\theta(\Gamma)| = |C|\,|\theta(\Gamma)|_L$ in the norm (3.31).

The proof of (3.30) is even simpler than that of BKL, (3.21), since the particle number estimates are not needed.

3.5. Proofs

The proofs will closely follow those of BKL, Theorem 6, with which we assume some familiarity. The only new ingredient here is the introduction of the temperature-dependent contours. This requires a new proof of the Peierls bound, which was given in Section 3.3.

In both Theorems A and B, one has two statements: One of them asserts the existence of Gibbs states which are small perturbations of the corresponding dominant ground states. The other is a uniqueness result: there are no extremal periodic Gibbs states other than the ones constructed. The proofs below, following refs. 3, 29, and 30, deal with the first statement only. There exist several proofs of the uniqueness result in the standard PS framework (refs. 16, 24, 28, 38; also Preiss, unpublished results), which can be presumably adapted to the present situation. An independent proof along the lines of the Appendix of ref. 4 is given in the Appendix here.

Proof of Theorem A. Clearly, this theorem may be deduced from Theorem B (set $\mu_0 = \tilde{\mu}_0 = 0$). However, since Theorem A can be proved without using contour models with parameters, we discuss this simple case

first. Actually, since all the ground states of $\mathscr{G}^*$ are equivalent, one could probably prove Theorem A without using contour models at all, in the spirit of refs. 17 and 18. However, since these symmetries are not necessarily pointwise ("spin flips"), but include lattice translations and rotations, it is simpler to use contour models. Thus, we want to find contour τ-functionals $F=(F_G)_{G\in\mathscr{G}^*}$ such that

$$Z(\Gamma; F)=Z(\Gamma) \qquad \text{for any contour } \Gamma \tag{3.33a}$$

with

$$s(F_G) \qquad \text{independent of } G \tag{3.33b}$$

[Uniqueness of solution of (3.33a) implies that $F_{G'}$ and $F_{G''}$ are related by the same symmetries of the Hamiltonian which relate G' and G'', and also that if F is a τ-functional, then $s(F_G)$ is G independent.] Once such functionals are found, the rest of the proof is standard.[29,30] Comparing (3.9) and (3.24), we obtain

$$Z^0(\Lambda \mid Y)=Z(\Lambda; F_{G(Y)} \mid Y)$$

Inserting this into (3.7), using (3.26) and taking logarithms, we obtain that the contour functionals must solve the equations

$$F_G(\Gamma)=\Psi(\Gamma)+\sum_{G'\in\mathscr{G}^*} \Delta(\mathrm{Int}_{G'}\Gamma; F_G \mid \Gamma)-\Delta(\mathrm{Int}_{G'}\Gamma; F_{G'} \mid \Gamma)$$

Using the fact that Ψ is a τ-functional (with $\tau=\rho$), which follows from Lemma 3.3 (with $\mu_0=0$), the Lipschitz bound (3.28), and the contraction mapping principle, one obtains a solution of (3.33a) in the Banach space defined by (3.22), satisfying (3.33b) and (3.23) with $\tau=\rho/2$.

Proof of Theorem B. We explain first how the proof may be reduced to BKL, Theorem 6. Then, for the sake of completeness and because it is not so easy to read off the precise region in the parameter space where our results hold from BKL, Theorem 6, we sketch the proof of the theorem.

The basic Hamiltonian H of BKL is the $H_{\mu_0(\beta)}$ here, for β large. With each dominant ground state we associate the restricted ensemble of its excitations of energy less than E. Thus, we have g restricted ensembles, each of them diluted (assumption A1 in BKL) by the estimates of Section 2.3. Note that in the definition of these restricted ensembles, the large scale does not appear yet. The Peierls Condition (assumption A2 in BKL) is defined by going to the large-scale lattice, as in Section 3.1. This is the major difference with respect to BKL. In BKL, Peierls Condition consists of two bounds: part (i) follows from our assumptions and Lemma 3.3, but

with $|\Gamma|$ of BKL replaced by $|\Gamma|_L$, the number of large-scale boxes in $\mathrm{supp}(\Gamma)$. We shall not need part (ii) of A2 of BKL. Assumption A3 is exactly our hypothesis that the free energies $f_E(\mu|G)$ are independent of $G \in \mathcal{G}^*$. The smoothness A4 of the perturbations is trivial here since the restricted ensembles clearly remain dilute for $\mu - \mu_0$ small. In particular, BKL, (2.8) holds [see (2.20) and (2.21), which is stronger, since we do not have ρ on the rhs]. Finally, our assumption that the perturbations remove the degeneracy of the ground states implies that BKL, (A5) holds, i.e., that the perturbative phase diagram is complete. The constants in BKL, Theorem 6 are related to the E and β here as follows: ε_0 measures the diluteness of the restricted ensemble and can be made as small as one wishes by taking β large. ρ is proportional to βE (see Lemma 3.3). Following the proof of BKL, as we do below, one sees that the η of BKL can be taken equal to $\alpha(\beta e^{\beta E_D})^{-3/2}$, for β and E large enough, which yields the proof.

Now we follow the proof of BKL. The recursion relations are essentially the same (except that they are defined on a large scale), and were discussed in Section 3.2. Equation (3.9) here replaces Lema 2 of BKL. As for Lemma 1 of BKL, it holds essentially unchanged, except for the change of scale again. Part (a) was discussed in Section 3.2 and part (d) is not necessary; the other two parts take the following form.

(b) There exists a c such that

$$\left| \frac{d}{d\mu} \psi_A \right| \leqslant c|\Gamma|$$

for any μ satisfying (2.22) and any $Y \in \mathcal{X}^{G,E}$. This follows from (2.18).

(c) For any μ satisfying (2.22),

$$\sum_{\mathrm{supp}(\Gamma) = M} e^{-\psi_A(\Gamma|Y)} \leqslant e^{-\rho|M|_L/2} \tag{3.34}$$

which is proved as in BKL, using the fact that μ satisfies (2.22) and

$$|C|\alpha(\beta e^{\beta E_D})^{-3/2} \leqslant \alpha$$

(3.34) shows that Ψ is a τ-functional with $\tau = \rho/2$, not only on $\mu_0(\cdot)$, but also in its neighborhood given by (2.22).

Now we use contour models with parameters. For each μ satisfying (2.22), we need τ-functionals F_G and parameters b_G, $G \in \mathcal{G}^*$, such that

$$Z(\Gamma) = e^{b_G v(\Gamma)} Z(\Gamma; F) \qquad \text{for any contour } \Gamma \tag{3.35a}$$

and

$$b_G = f_E(\mu|G) - s(F_G, \mu) + \alpha \tag{3.35b}$$

where α is determined by the condition that $\min_G b_G = 0$. These equations are equivalent to

$$F_G(\Gamma) = \Psi(\Gamma) + \sum_{G' \in \mathscr{G}^*} \{\Delta(\mathrm{Int}_{G'}\Gamma; F_G, \mu \mid \Gamma) - \Delta(\mathrm{Int}_{G'}\Gamma; F_{G'}, b_{G'}, \mu/\Gamma)\}$$

which can again be solved by the contraction mapping principle in the Banach space defined by (3.31) using the Lipschitz bounds (3.30) (see the proof of Theorem 6 in BKL for more details). As in the proof of Theorem A, the only crucial point (apart from the estimates on the contour models) is the fact that Ψ is a τ-functional, which follows from Lemma 3.3 and (3.31). The phase diagram for μ satisfying (2.22) is given by the map $\mu \mapsto t(\mu)$, $t(\mu)_G = b_G$ (b_G depends on μ!), which is Lipschitz by (2.18) and (3.32). That the inverse map is Lipschitz, too, is also proved as in BKL.

Remarks to the Proofs of the Extensions of Theorem B. We need to consider remarks 4 and 6 only. Remark 4 holds because, wherever we have local domination and equality of the free energies $f_E(\mu \mid G)$ for G in $\mathscr{G}^*$, which here follows from the equivalence of the ground states of $\mathscr{G}^*$, Lemma 3.3 holds, and this shows that Ψ is a τ-functional. Then one may simply repeat the proof of Theorem A: from the definition (2.9) and local domination and the bound (2.18), it is obvious that if local domination holds in $\mathscr{K}$, it holds in a neighborhood of it (possibly with a smaller c, uniformly in the region $\mathscr{K}$).

Remark 6 is somewhat more delicate, but there are only two points to check: The first thing to show is that the perturbative phase diagram is complete, since the first term in the rhs of (2.17) may now vanish. This, however, still follows from the inverse function theorem, for μ satisfying (2.22), because we assume that A, which is the matrix of partial derivatives of $f_E(\mu \mid G)$ with respect to μ, is invertible and that $|\mu - \mu_0|$ is so small that

$$|\mu - \mu_0| \cdot \|A^{-1}\| \ll 1 \qquad \text{for } \beta \text{ large}$$

The second point is that the Lipschitz constant for t^{-1} (denoted L in BKL, not to be confused with the spacing L of our large-scale lattice) is of order $e^{c\beta}$. This could cause a problem, since the first inequality of BKL, (4.5), which is used in the proof of the invertibility and Lipschitz continuity of t, could fail. However, we assume here that $f_E(\mu \mid G)$ is independent of $G \in \mathscr{G}^*$ for E large. Looking at the proof of Lemma 3.3, we see that ρ, to which τ is proportional, can be made large by choosing E large (more precisely, one defines $L = (\beta E e^{-\beta E_D})^{1/2}$, note the E, and the same proof gives now $\rho = c_1 \beta E$). Then E can be chosen so large that the first inequality of BKL, (4.5), holds. The intuition here is that since $\|A^{-1}\|$ is large, one can control only

a small neighborhood of $\mu_0(\cdot)$ (the first point above). But then one needs to know that the true line of coexistence of all the phases corresponding to the ground states of $\mathscr{G}^*$ falls into that neighborhood. This is obtained by choosing E large, since the larger the E, the closer one gets to the true coexistence line.

4. EXAMPLES: PROOF OF THE RETOUCH PROPERTY AND PEIERLS CONDITION

We now come back to the models considered in Section 1. In all these models, whenever $\mathscr{G}$ is infinite and $\mathscr{G}^*$ finite, the ground states have *layered structure*, in the sense that the sets $\Lambda'(G)$ of Condition $(\mathscr{L})$ of Section 3.3 consist of a number of planes.

In one example—the Balanced Model—the proof of the retouch property and calculation of the phase diagram are given in detail. Since the retouch property of the fcc model is proved in exactly the same way, we skip the argument. The low-temperature expansion for the fcc model has been treated in a number of works. In Section 4.2 we comment on the relation of these earlier calculations to ours. In particular, we show how deduce an effective m-potential from calculations involving periodic ground states only. Part of the low-temperature phase diagram of the three-dimensional ANNNI model is treated in Section 4.3. The retouch property of this model is the simplest to prove, since it follows easily from a general result on "stacked models" in the next section. Finally, in the last section we return to the stacked antiferromagnet of Section 1.4.

In this section, unlike in the preceding one, by a cube we understand an elementary cube of the cubic lattice.

4.1. The Balanced Model

Global ground states of the model and part of its phase diagram have been discussed in the Introduction. To analyze the local ground states, it is useful to consider the potential Φ for the Hamiltonian (0.2) obtained by grouping its terms within the plaquattes: Let $P = \{a, b, c, d\}$ be a plaquette of the lattice, b, c, d, a being nearest neighbors of a, b, c, d, respectively; then

$$\Phi_P = -\tfrac{1}{2}(\sigma_a\sigma_b + \sigma_b\sigma_c + {}_c\sigma_d + \sigma_d\sigma_a) + \tfrac{1}{2}(1+\mu)(\sigma_a\sigma_c + \sigma_b\sigma_d) \quad (4.1)$$

We have

$$\Phi_P\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right) = \Phi_P\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right) = -1+\mu, \qquad \Phi_P\left(\begin{smallmatrix}+&-\\+&-\end{smallmatrix}\right) = -1-\mu \quad (4.2)$$

$$\Phi_P\left(\begin{smallmatrix}+&+\\+&-\end{smallmatrix}\right) = 0, \qquad \Phi_P\left(\begin{smallmatrix}+&-\\-&+\end{smallmatrix}\right) = 3+\mu \quad (4.3)$$

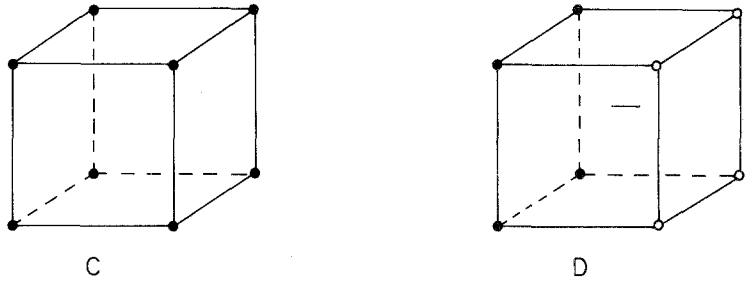

Fig. 4. Ground-state cubes of the Balanced Model.

with the obvious notation. Thus, for $\mu = 0$ the ground states of the Introduction are in the $\mathcal{G}(\Phi)$ of Section 2.1, and therefore Φ is an m-potential. Furthermore, it is easy to see that the only cubes with all plaquettes of type (4.2) are those of Fig. 4. This implies that the ground states of the Introduction exhaust $\mathcal{G}(\Phi)$. According to the definition of Section 2.1, a configuration G on Λ is a (local) ground state for $\mu = 0$, i.e., an element of $\mathcal{G}_\Lambda(\Phi)$, if $\Phi_P(G) = -1$ for any plaquette P contained in Λ. Ground states of a cube are pictured in Fig. 4. The full dot stands for plus and the small circle for minus. The first of the configurations and the one obtained from it by global spin flip appear in the dominant ground states of type C of the Introduction; cubes with such configurations are called C-cubes. Cubes obtained by rotation of the other cube of Fig. 4 are called D-cubes—they appear in the ground states of type D of the Introduction.

We prove now the retouch property. Consider parallelepipeds formed by unions of the cubes of the lattice. By inspection of Fig. 4, one can see that ground states of any parallelepiped are such that either (i) each cube of the parallelepiped is a C-cube, or (ii) one of the cubes of the parallelepiped is not a C-cube, in which case there is a uniquely defined coordinate axis such that the intersection with the parallelepiped of each plane perpendicular to it consists of cubes of the same kind. The direction of this axis—marked by a dash at the center of the cubes of Fig. 4—is called the *soft direction* of the parallelepiped.

We note that, obviously, it is enough to prove the retouch property for the elementary excitations. Thus, let us fix an energy E and let X be an elementary excitation with $H_0(X) < E$. We can choose l large enough so that each of the components of the support of such an excitation X is contained in a parallelepiped which in turn is contained in the interior of another parallelepiped contained in $\mathrm{dom}(X)$, as in Fig. 5. All the cubes of the larger parallelepiped not contained in the smaller one are ground-state cubes.

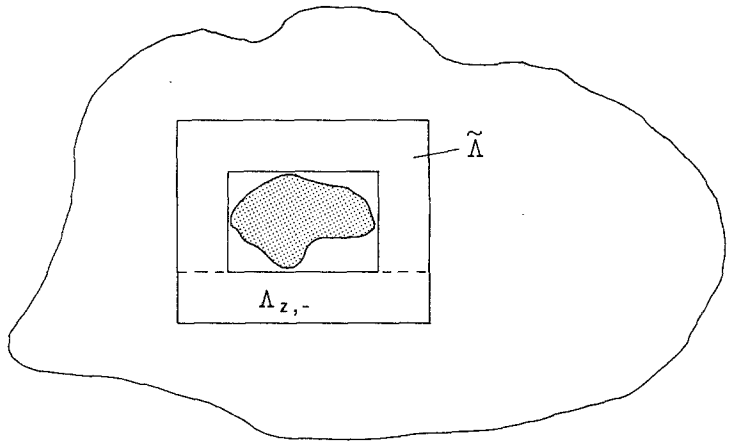

Fig. 5. Diagram for the proof of the retouch property of the Balanced Model.

Consider now a domain Λ consisting of a parallelepiped $\tilde{\Lambda}$ with a cut-out smaller parallelepiped (Fig. 5). Λ is a union of six nonempty overlapping parallelepipeds $\Lambda_{x,\pm}$, $\Lambda_{y,\pm}$, $\Lambda_{z,\pm}$. If in a ground state of Λ there are C-cubes only, then since the parallelepipeds $\Lambda_{i,\pm}$ overlap, condition (i) implies that the cubes are of the same kind. Hence, obviously, in this case one can extend the ground state of Λ to a ground state of $\tilde{\Lambda}$ in a unique way.

If not all the cubes of Λ are C-cubes, then one can assume without loss of generality that $\Lambda_{z,+}$ contains a D-cube. If z is not the soft direction of $\Lambda_{z,+}$, then the proof is trivial. If z is the soft direction of $\Lambda_{z,+}$, then by (ii) any plane perpendicular to the z axis which intersects Λ consists of cubes of the same kind. In particular, intersections of $\Lambda_{z,+}$ with $\Lambda_{x,i}$ and $\Lambda_{y,i}$ contain only translations of the D-cube above and, again by (ii), intersection of $\Lambda_{x,i}$ and $\Lambda_{y,i}$ with any plane perpendicular to the z axis consists of cubes of the same kind, cubes for which z is the soft direction. It is now easy to see that the intersection of Λ with any plane perpendicular to the z axis consists of cubes of the same kind and that therefore one can extend the configuration to a ground state of $\tilde{\Lambda}$ in a unique way. Note that one obtains here also a proof of the fact that ground states have unique extensions from the boundaries, as required by the retouch property.

We now turn to a discussion of the excitations and of local domination. We proceed as described in Section 2.4. Excitations up to order three are pictured in Fig. 6. For models which are not ferromagnetic, like the present model or the models of the next section, a list of excitations is usually not accompanied by a proof that it is exhaustive. Instead, one computes energies of excitations obtained by flipping a number of spins

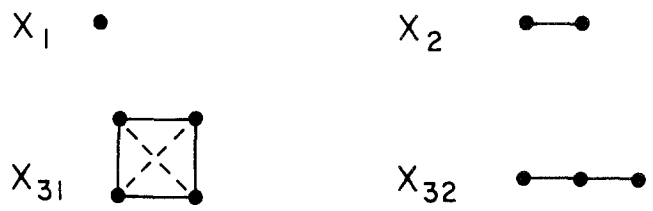

Fig. 6. Excitations of the Balanced Model, up to third order; $E_1 = 12$, $E_2 = 16$, $E_3 = 20$. Excitations differing by overall spin flip are denoted by the same symbol.

and hopes that no excitation of low order is missed. Since in the expression for the energy of an excitation in terms of the natural potential one has both positive and negative terms, cancellations usually make it hard to see if flipping more spins does not yield excitations of lower energy. However, if an m-potential is used, estimations are very much like in the ferromagnetic case, as can be seen in the following sketch of a proof of the completeness of the list of Fig. 6.

Consider the m-potential Φ of (4.1) with $\mu = 0$. It is easy to see that any excitation has at least 12 excited plaquettes, i.e., plaquettes of type (4.3). Therefore, $E_1 = 12$; and since $20 < 2E_1$, it is enough to consider Φ-connected excitations (in the obvious sense). We assume that it has been already checked that X_1, X_2, X_{31} are the only excitations of a unit cube with energy $\leqslant 20$, with the help of a computer, as we did, or otherwise. Let X be a Φ-connected excitation with support not contained in a unit cube, and let

$$\alpha = \min\{x_1 : x \in \mathrm{supp}(X)\}, \qquad \beta = \max\{x_1 : x \in \mathrm{supp}(X)\}$$

We can assume that $\beta - \alpha \geqslant 2$ and that each plane $P_\gamma = \{x: x_1 = \gamma\}$ with $\alpha \leqslant \gamma \leqslant \beta$ intersects $\mathrm{supp}(X)$. We note that each such plane contains four or more plaquettes intersecting $\mathrm{supp}(X)$ at exactly one point. Furthermore, there are at least four plaquettes perpendicular to P_α contained in $\{x: x_1 \leqslant \alpha\}$ intersecting $\mathrm{supp}(X)$ at exactly one point. And similarly for P_β. This yields at least $4 \cdot 3 + 8$ excited plaquettes, and energy $\geqslant 20$. In fact, it is easy to see that if P_γ contains more than one point of $\mathrm{supp}(X)$ for some γ, then the number of excited plaquettes is strictly larger than 20. Thus, X_{32} is the only excitation with energy 20.

We define now an effective potential (2.5) for excitations up to third order. For a local ground state G and a point a of the lattice we set

$$f_{(3)}(G, a) = \beta \sum_{P \ni a} \tfrac{1}{4}\Phi_P(G) - n_1(G, a)e^{-\beta E_1} - \tfrac{1}{2}n_2(G, a)e^{-\beta E_2} - [\tfrac{1}{4}n_{31}(G, a) + n_{32}(G, a)]e^{-\beta E_3} \tag{4.4}$$

where $n_\alpha(G, a)$ ($\alpha = 1, 2, 31$) is the number of excitations to type X_α of the ground state G whose support contains a and $n_{32}(G, a)$ is the number of excitations of G of type X_{32} for which a is the *middle point* of the support of the excitation. This can be obtained by choosing the weight function $\chi(a, \vartheta)$ to be standard unless $\vartheta = \vartheta_{X_{32}}$ and $\chi(a, \vartheta_{X_{32}}) = 1$. The function $\chi(a, B)$ is also standard. Since only m.f.'s of excitations are involved here (i.e., $\vartheta = \vartheta_X$), $\phi^T(\vartheta) = \phi(\vartheta)$ in (4.4). Furthermore, anticipating the final result, we set $\mu = 0$ in the coefficients of $e^{-\beta E_1}$, $e^{-\beta E_2}$, and $e^{-\beta E_3}$. For, it is easy to see that in case the perturbation (here $\frac{1}{2}\mu \sum_{\text{n.n.n.}} \sigma_a \sigma_b$) removes the degeneracy of the dominant ground states, retaining μ in the exponents yields a correction of order at least $2E_1$, which is here larger than E_3. The $n_\alpha(G, a)$ depend on the restriction of G to $[a, 1] = \{x: |x - a| \leqslant 1\}$ only. This dictates $l \geqslant 1$ in the definition of excitations and the retouch property.

According to the above description of ground states, the restriction of G to $[a, 1]$ is obtained from the plus ground state by choosing one of the coordinate axes and flipping all spins in a number of planes perpendicular to this axis. Obviously, $n_1(G, a) = 1$ for each G and a. The other n_α are expressed conveniently through the *structural constants* n_C and n_{CC}: Let $n_C(G, a) = 0$ if there are no neighboring planes of the same sign intersecting $[a, 1]$, $n_C(G, a) = 1$ if there is one pair of such planes, and let $n_C(G, a) = 2$ otherwise. Furthermore, let $n_{CC}(G, a) = 1$ if all the spins of $[a, 1]$ are of the same sign, and $n_{CC}(G, a) = 0$ otherwise. Then $n_2(G, a) = 4 + n_C(G, a)$, $n_{31}(G, a) = 4 + 2n_C(G, a)$, $n_{32}(G, a) = 1 + n_{CC}(G, a)$, and, by (4.2) and (4.3), the sum over plaquettes in (4.4) is equal to

$$(-1 + \mu) + \tfrac{1}{2} n_C(G, a)(-1 + \mu) + \tfrac{1}{2}(-1 - \mu)[1 - n_C(G, a)]$$

Thus,

$$f_{(3)}(G, a) = \beta\{-\tfrac{3}{2} + [\tfrac{1}{2} + n_C(G, a)]\mu\} - e^{-\beta E_1} - \tfrac{1}{2}[2 + n_C(G, a)]e^{-\beta E_2} - [2 + \tfrac{1}{2} n_C(G, a) + n_{CC}(G, a)]e^{-\beta E_3} \tag{4.5}$$

which for $\mu = 0$ yields

$$f_{(2)}(G, a) = -\tfrac{3}{2}\beta - e^{-\beta E_1} - \tfrac{1}{2}[2 + n_C(G, a)]e^{-\beta E_2}$$

Obviously, $f_{(2)}(G, a)$ is minimal for $n_C(G, a) = 2$, which corresponds to all spins within $[a, 1]$ being of the same sign. Patching these local ground states, we obtain that the C-ground states are strongly dominant in order 2 (Section 2.4), and that by Theorem A there are exactly two pure phases at low temperatures, as described in the Introduction. We now consider μ small but nonzero.

We note that for the C-ground state $n_C = 2$, $n_{CC} = 1$, and for the D-ground state $n_C = n_{CC} = 0$. Thus, from $f_{(3)}(G^C, a) = f_{(3)}(G^D, a)$ one obtains

$$K_{(3)}^{C,D} \equiv \tfrac{1}{2} e^{-\beta E_2} + e^{-\beta E_3} \tag{4.6}$$

where $K = \beta\mu$. The $K_{(3)}^{C,D}$ is marked by a dashed line in Fig. 7.

The behavior of the system in a neighborhood of the line (4.6) is seen most clearly if one defines $\hat{f}_{(3)}$ by first substituting $K = K_{(3)}^{C,D} + \hat{K}$ and then by skipping in (4.5) terms which do not depend on n_C and n_{CC}:

$$\hat{f}_{(3)}(G, a) = n_C(G, a)\hat{K} + [\tfrac{1}{2} n_C(G, a) - n_{CC}(G, a)] e^{-\beta E_3} \tag{4.7}$$

f and $\hat{f}$ yield the same phase diagrams. Since $n_{CC} = 1$ for $n_C = 2$, and $n_{CC} = 0$ otherwise, one obtains that G^C is dominant for $\hat{K} \leqslant 0$ and G^D for $\hat{K} \geqslant 0$. According to Theorem B and Remark 4 of Section 2.5, there exists a line $\beta \mapsto K^{C,D}(\beta)$ defined for β large enough (marked bold in Fig. 7) such that

$$K^{C,D}(\beta) - K_{(3)}^{C,D}(\beta) = O(e^{-\beta E_4})$$

with the property that for $K > K^{(3)}(\beta)$ one has exactly six pure phases (the D-phases), for $K < K^{(3)}(\beta)$ one has the two C-phases, and on the line $K = K^{(3)}(\beta)$ one has eight pure phases.

Remark. Since $\hat{f}_{(2)}$ is homogeneous in n_C, in second order all phases coexist on the line $K = K_{(2)}^{C,D}$ defined by $\hat{f}_{(2)}(G^C) = \hat{f}_{(2)}(G^D)$. Thus, one has to go to higher order to see which ground states dominate and to obtain the phase diagram. One has a similar situation in other models.

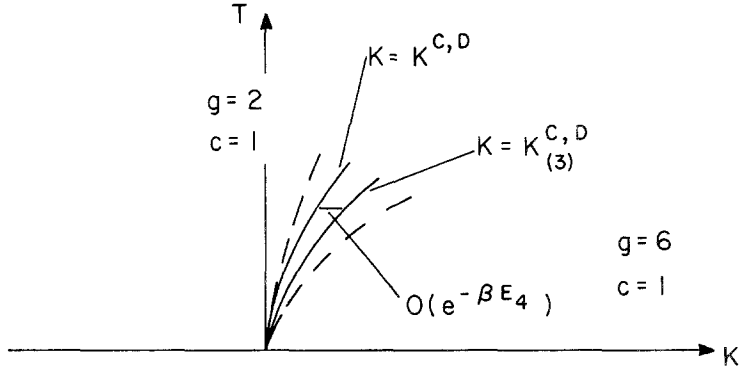

Fig. 7. Low-temperature phase diagram of the Balanced Model.

4.2. Antiferromagnet on Face-Centered Cubic Lattice

The ground states of (1.3) have been determined by Danielian.[6] We need a refinement of these results—their localized version.

On an α-cubic of Fig. 8 we connect points of a *tetrahedron* of the lattice. There are four such tetrahedra in each cube, and every pair of nearest neighbors is contained in two tetrahedra. We will use the potential Φ of the unperturbed system obtained by grouping the terms of (1.3) within the tetrahedra:

$$\Phi_B = \frac{1}{2} \sum_{\{a,b\} \subset B} \sigma_a \sigma_b \tag{4.8}$$

If B is a tetrahedron of the lattice, where the sum is over (unordered) pairs, and $\Phi_B = 0$ otherwise. This is an m-potential, since Φ_B is minimal for configurations of the tetrahedron with two pluses and two minuses, and the global configurations described in Section 1.2 are of this type.

A configuration G on Λ is in $\mathcal{G}_\Lambda(\Phi)$ if $\Phi_B(G) = -1$ for any tetrahedron B contained in Λ. Ground states of a cube are pictured in Fig. 8. The full dot stands for plus and the small circle for minus. The soft directions, in the sense of the preceding section, are marked by a dash on each of the ground-state cubes. The first of the configurations and those obtained from it by global flip and rotations appear in the dominant ground states, and are said to be of *type* β.

Having the ground-state cubes and their "propagation properties," the retouch property is proved here in exactly the same way as in the case of the Balanced Model. We now turn to a discussion of the excitations and to

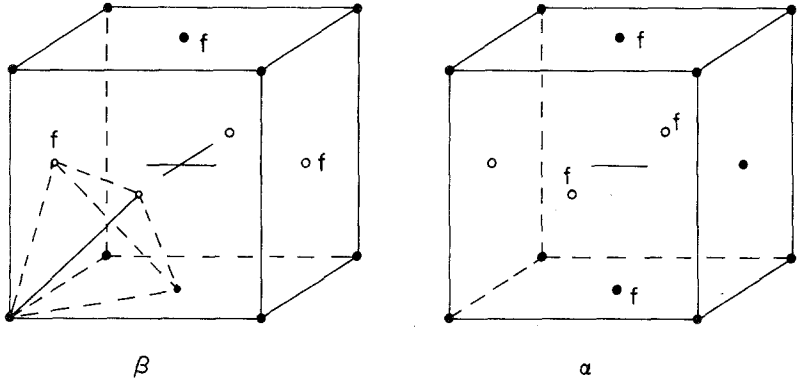

Fig. 8. Ground states of an elementary cube of the antiferromagnet (1.3) on the fcc lattice.

a proof of the local domination. We first give the usual derivation of the perturbation expansion, as encountered in theoretical physics, then pass to the localized version which is needed here (see also Chapter III of ref. 32). Excitations up to order three are pictured in Fig. 9.

The usual derivation has two versions, called *primitive method* and *cumulant method* in ref. 45. The cluster expansion (2.4) can be considered as a general version of the cumulant method. We start with the primitive method, and then compare it with expansion based on (2.4), which allows us, in particular, to comment on its (in)dependence of the choice of the constant l which appears in the definition of excitations and the retouch property.

We consider two periodic ground states G' and G'', and write Z' for $Z_R(\Lambda \mid G')$ and Z'' for $Z_R(\Lambda \mid G'')$. Obviously, excitations of order 1 yield the same contribution to Z' and Z''. The same applies to excitations of order 2. Consider now the excitation X_{31} (and X_{32}) of Fig. 9 (C_{31} of ref. 6), in which two noninteracting, i.e., non-n.n., spins of the same sign are flipped. In the primitive treatment one would compute the number of excitations of this type in a system of N spins, which is $(N/2-5)N/2$, disregarding a boundary term. The -5 comes from the fact that once the position of one of the flipped spins is fixed, the other spin is not allowed to occupy this position and the neighboring sites. When $\log Z$ is computed, the term quadratic in N cancels out, and on dividing by N one obtains the contribution $-5/2$ to the free energy. Now, in the computation of $\log Z$ based on the cluster expansion (2.2) and (2.4) the terms quadratic in N have been taken care of already while the term $-5/2$ is obtained from two kinds of multiplicity functions. For, depending on the choice of the constant l of the definition of elementary excitations, a ground state with two flipped plus spins can be a single elementary excitation or can consist of two elementary excitations. These would yield in $\log Z$ terms of the same magnitude but of opposite signs. Cancellation between these two kinds of

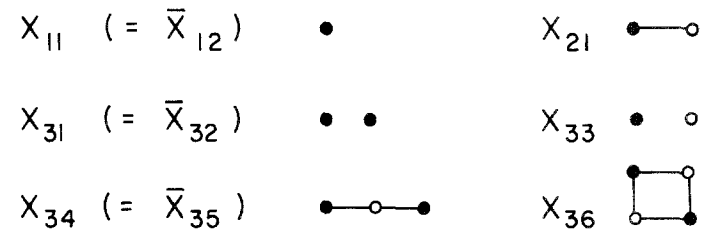

Fig. 9. Excitations of the model antiferromagnet (1.3) on the fcc lattice, up to third order[6]; $E_1 = 8J$, $E_2 = 12J$, $E_3 = 16J$. Excitation of type X_{34} is marked by f's on the α-cube of Fig. 8 and an excitation of type X_{36} on the β-cube. $\bar{X}_{ij}$ is obtained from X_{ij} by flipping all the spins.

contributions is responsible for the fact that the sum of terms of order 3 does not depend on the choice of the constant l, and that in fact what remains after performing these cancellations is a sum over multiplicity functions ϑ which are equal to 1 on pairs of excitations of type X_{11} for which the flipped points "interact," i.e., are either nearest neighbors or coincide. In this way the term $-5/2$ is recovered. Similar considerations apply to the excitation X_{33} or to any disconnected excitation, like the fourth order X_{42} which is considered further on.

All this implies that multiplicity functions of type X_{31} and X_{33} yield the same contributions to $\log Z'$ and $\log Z''$: each contributing pair of points is within a tetrahedron and all the ground-state tetrahedra yield the same contributions.

The domination of the six ground states containing only the β-cubes shows up in the third order, when one considers excitations of type X_{36}: The ground state configurations contain α- and β-cubes only. By direct enumeration one can see that there are as many excitations of type X_{34} and X_{35} of an α-cube as there are of a β-cube. Furthermore, obviously there is one excitation of type X_{36} of an α-cube and there are two of a β-cube. Since no excitation of type X_{36} is shared by two cubes, their contribution to $(\log Z' - \log Z'')$ is equal to (number of β-cubes in G' − number of β-cubes in $G'') \times e^{-\beta E_3}$. Thus, since the β-ground states have maximal number of β-cubes, their domination is established.

We will now amplify these considerations by deducing from them the effective m-potential and then use the standard perturbative calculations to obtain the complete phase diagram.

The above discussion suggests we define the effective potential as follows: the index a in $f_{(3)}(G, a)$ will standard for the center of one of the elementary cubes of the fcc lattice. The m.f.'s appearing in $\tilde{f}_{(3)}(G, a)$ will all have supports contained in the corresponding cube, and the weights will be standard. Thus,

$$e_G(a) = \frac{1}{4} \sum_{B \subset \text{unit cube centered at } a} \Phi_B(G)$$

and

$$\begin{aligned}\tilde{f}_{(3)}(G, a) = &- [\chi(11)\, n_{11}(G, a) + \chi(12)\, n_{12}(G, a)]\, e^{-\beta E_1} - \chi(21)\, n_{21}(G, a) e^{-\beta E_2} \\ &+ [\chi(31)\, n_{31}(G, a) + \chi(32)\, n_{32}(G, a) + \chi(33)\, n_{33}(G, a)]\, e^{-\beta E_3} \\ &- [\chi(34)\, n_{34}(G, a) + \chi(35)\, n_{35}(G, a) + \chi(36)\, n_{36}(G, a)]\, e^{-\beta E_3}\end{aligned}$$

Since, as follows from the above discussion, all $n_{ij}(G, a)$ are G independent, apart from n_{36}, we need to compute only the latter quantity to check for

the domination. Now, again from the above discussion, $\chi_{34} = 1/3$, $\chi_{36} = 1$, whereas

$$n_{34}(G^\alpha) + n_{35}(G^\beta) = n_{34}(G^\beta) + n_{35}(G^\beta), \qquad n_{36}(G^\alpha) = 1, \qquad n_{36}(G^\beta) = 2$$

Thus, G^β is dominant in third order. We pass now to the perturbation (1.4) of the model, with $h = 0$. Again, considering the perturbed Hamiltonian and setting $f_{(3)}(G^\alpha, a) = f_{(3)}(G^\beta, a)$ one obtains a line $\alpha_{(3)}(\beta)$ on which in the third order the α- and β-cubes "coexist," and therefore one sees an infinite number of phases in this order. Thus, one has to consider excitations of the next order to find the low-temperature phases in a neighborhood of this line (Fig. 10). Instead of working out the effective potential *ab initio*, we will show how to use standard perturbative calculations involving periodic ground states only, to deduce the existence of an effective m-potential, and hence to obtain a proof of local domination. We reformulate first somewhat the calculations of ref. 23.

Let G be a periodic ground state of the system. One can assume without a loss of generality that the soft direction of G is x. A *plane* is a plane which is perpendicular to the x axis and which is passing through points of $\mathbb{L}$. Following ref. 23, a plane P will be called a *β-plane* (resp. an *α-plane*) if all the cubes of the lattice with centers in P are β-cubes (resp. α-cubes). According to the description of ground states given in Section 1.2, each plane is either an α-plane or a β-plane. Let n_α and n_β be the density of α- and β-planes, respectively. That is, if N is a multiple of the period of G and one has a sequence of N consecutive planes, then

$$n_\beta(G) = (\text{number of planes of type } \beta \text{ in the sequence})/N$$

Similarly, let $n_{\beta\beta}(G)$ be the density of pairs of consecutive β-planes in G and let the *structural constants* $n_{\alpha\alpha}$, $n_{\alpha\beta}$, and $n_{\beta\alpha}$ (in the sense of ref. 12) be

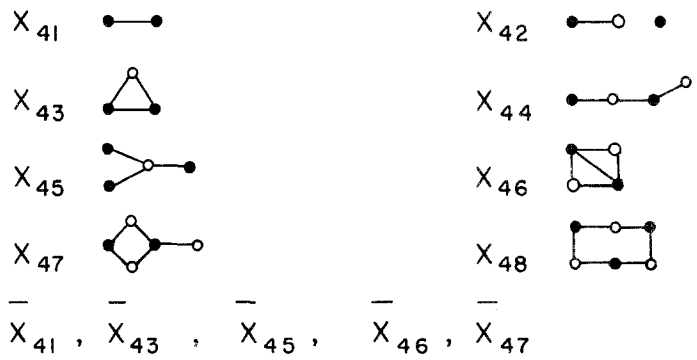

Fig. 10. Fourth-order excitations of the model (1.3); $E_4 = 20J$.

defined analogously. According to ref. 23, one has the low-temperature expansion

$$-f_{(E_4)}(G) = 2\beta - [1 + 2n_\beta(G)]K + x^2 + 4x^3 + [\tfrac{29}{2} + n_\beta(G)]x^4 + [64 + 8n_\beta(G) + 2n_{\beta\beta}(G)]x^5 \tag{4.9}$$

Here $K = \beta\alpha$ ($= K_2$ of ref. 23), and we set $x = e^{-4\beta}$, so that $e^{-\beta E_i} = x^{i+1}$ for $i \geqslant 1$, as in ref. 23. We also set $K = 0$ in the coefficients of x^k in anticipation of the fact that $K = O(x^4)$ in the region of interest here. Skipping in (4.9) the G-independent terms, we consider the $\hat{f}(G)$ defined by

$$-\hat{f}(G) = -2n_\beta(G)K + n_\beta(G)x^4 + [8n_\beta(G) + 2n_{\beta\beta}(G)]x^5$$

Setting $K = 0$, as before, one can see that minimal free energy is obtained when n_β is minimal, i.e., for the β-ground states, and that these ground states should dominate in order 3. Also, the same ground states dominate for $K < 0$. For $K > 0$, β-ground states dominate in the same order untill

$$K = K_{(3)}^{\alpha,\beta} \equiv \tfrac{1}{2}x^4 \tag{4.10}$$

On the curve (4.10) in order 3 all ground states will appear, since the coefficient of n_β is zero. Thus, to investigate domination in a neighborhood of (4.10), one has to pass to the next order. Setting $\hat{f}(G^\alpha) = \hat{f}(G^\beta)$, one obtains

$$K = K_{(4)}^{\alpha,\beta} \equiv \tfrac{1}{2}x^4 + 5x^5$$

Thus, on the curve $K = K_{(4)}^{\alpha,\beta}$

$$\hat{f}(G^\alpha) = 2[n_\beta(G) - n_{\beta\beta}(G)]x^5 = 2n_{\beta\alpha}(G)x^5$$

which is zero for $G = G^\alpha$ or $G = G^\beta$, and is strictly positive otherwise. This completes the argument. (A mistake in the coefficient of x^6 in ref. 23, which is of no relevance here, is corrected in ref. 33.)

To obtain from these calculations an effective m-potential, we proceed as follows. We write $f_{(4)}$ instead of $f_{(E_4)}$. We consider a translation-invariant family M_a, $a \in \mathbb{L}$, of subsets of the lattice with the property that if $f_{(4)}(G, a)$ are defined by standard weights χ [with respect to the family (M_α)], then $f_{(4)}(G, a)$ is equal to $f_{(4)}(\tilde{G})$ for a global extension $\tilde{G}$ of G. This will imply that on the curve (4) the α- and β-ground states are dominant and that $f_{(4)}(G, a)$ is an effective m-potential.

Since the figures M_a may be hard to visualize, we give their coodinate description. First, we realize the fcc lattice $\mathbb{L}$ as

$$\mathbb{L} = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^i \in \frac{1}{2} \cdot \mathbb{Z}, \sum_i x^i \in \mathbb{Z}\right\}$$

For each lattice point a, let C_a be the following face-centered cube with a corner at a:

$$C_a = \{x \in \mathbb{L}: x^i \geqslant a^i,\ i = 1, 2, 3, \text{ and } \max_i |x^i - a^i| \leqslant 1\}$$

Furthermore, let $f_1 = (0, \frac{1}{2}, \frac{1}{2})$, $f_2 = (\frac{1}{2}, 0, \frac{1}{2})$, $f_3 = (\frac{1}{2}, \frac{1}{2}, 0)$; so that $\{0, f_1; f_2, f_3\}$ is a tetrahedron B of (4.7). We define an *M-figure* as a set of the form

$$M_a = C_a \cup (C_a + f_1) \cup (C_a + f_2) \cup (C_a + f_3) \tag{4.11}$$

for some $a \in \mathbb{L}$; here $C_a + x$ is a translation of the cube C_a by a vector x. M_a consists of the points of the lattice which have all the coordinates larger than that of a and which are nearest neighbors of points of C_a. The definition (4.11) better exhibits the symmetries of M_a, which are needed below, than the verbal one.

Now, by inspection, it is clear that each excitation of Figs. 9 and 10 is contained in an M-figure. Furthermore, it follows from the description of local ground states that if G is a ground state in a cubic neighborhood of M_a, then we can classify M_a as being either of type $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, or $\beta\beta$, in direction of increasing "soft coordinates" (in the obvious sense). Let now $f_{(4)}(G, a)$ be defined by the standard weights $\chi(a, \cdot)$ with respect to M_a. Then $f_{(4)}(G, a)$ depends only on the restriction of G to M_a. Therefore, it is equal to $f_{(4)}(\hat{G}, a)$, where $\hat{G}$ is an extension of the restriction of G to M_a which is periodic in the soft direction of G, and which is defined as follows: If M_a is of type $\alpha\alpha$ or $\beta\beta$ for G, then $\hat{G}$ is defined as G^α or G^β, respectively. If M_α is of type $\beta\alpha$, then $\hat{G}$ is defined as the $G^{\beta\alpha} = \ldots\alpha\beta\alpha\beta\ldots$ ground state; it is not hard to see that this can be done. Similarly for the $\alpha\beta$ case.

We note now that $f_{(4)}(G, a)$ is the same in the $\alpha\beta$ and $\beta\alpha$ cases (it is here that we had to choose the shape of M_α with some care). Since all the M-figures of $G_{\alpha\beta}$ are either of type $\alpha\beta$ or $\beta\alpha$, we see that $f_{(4)}(G, a) = f_{(4)}(G^{\alpha\beta})$ whenever the restriction of a local ground state to M_α is of type $\alpha\beta$ or $\beta\alpha$. And obviously, when this restriction is of type α or β, then $f_{(4)}(G, a)$ is equal to $f_{(4)}(G^\alpha)$ or $f_{(4)}(G^\beta)$, respectively. Since on the line (4)

$$f_{(4)}(G^\alpha) = f_{(4)}(G^\beta) > f_{(4)}(G^{\alpha\beta})$$

one obtains that $f_{(4)}(G, a)$ is an effective m-potential and that the α- and β-ground states are dominant there. Moreover, the same argument shows that the β-ground states are dominant for $K < K_{(4)}^{\alpha,\beta}$ and the α-ground states for $K > K_{(4)}^{\alpha,\beta}$. Theorem B, together with Remark 4, implies now that there is a curve $K^{\alpha,\beta}(\beta)$ defined for large enough β, such that

$$K^{\alpha,\beta}(\beta) - K_{(4)}^{\alpha,\beta} = o(e^{-\beta E_4})$$

with the property that for $K > K^{\alpha,\beta}(\beta)$ there are exactly 12 pure phases, for $K < K^{\alpha,\beta}(\beta)$ there are 6, and for $K = K^{\alpha,\beta}(\beta)$ there are 18.

Similar calculations yield the same phase diagram in the neighborhood of the (open) interval OE of Fig. 5 of ref. 32. In fact, using the patching of the Remark 4 of Section 2.5, one obtains low-temperature phases for all points left to the segments AB and BE of Fig. 5 of ref. 32. One can analyze similarly the segment ED of that figure, apart from the fact here that one has three types of ground-state cubes, two different types appearing in the same global ground states. The low-temperature properties of the remaining part of Fig. 5 of ref. 32 have not been analyzed.

4.3. The ANNNI Model

That the model has the retouch property follows from the general arguments of the next section. We start the analysis of the model with a determination of the ground states—a computation which is a localized version of that of ref. 12. We keep the notation as close as possible to that of ref. 12.

A natural potential for (1.5) is a sum the usual n.n. potential for the first sum in (1.4) and the potential containing the terms of (1.5) within vertical three-point sets:

$$\Phi_{\{a-\hat{z},a,a+\hat{z}\}} = \Phi_a^{\text{ver}} = -\tfrac{1}{2}J_1(s_{a-\hat{z}}s_a + s_a s_{a+\hat{z}}) - J_2 s_{a-\hat{z}} s_{a+\hat{z}}$$

where $\hat{z}$ is the unit vector in the vertical direction. In the obvious notation, we have

$$\Phi_a^{\text{ver}}(+++) = -J_1 - J_2, \qquad \Phi_a^{\text{ver}}(++-) = J_2, \qquad \Phi_a^{\text{ver}}(+-+) = J_1 - J_2$$

Φ^{ver} will be identified with the corresponding potential of the one-dimensional system, which results in the notation that follows. Hence, $\Phi_a^{\text{ver}}(+++) = -J_1 + \Phi_a^{\text{ver}}(+-+)$, and $\Phi_a^{\text{ver}}(+++) = \Phi_a^{\text{ver}}(++-)$ yields $J_1 = -2J_2$. The case of $J_1 < 0$ is obtained from that of $J_1 > 0$ by flipping every second spin, i.e., spins in every second horizontal plane of the three-dimensional model. Therefore we will consider only the case of $J_1 > 0$ (for $J_1 = 0$ one has two decoupled Ising models). Taking the symmetries of Φ_a^{ver} into account and setting $\kappa = -J_2/J_1$, one obtains that Φ_a^{ver} is minimized by $(+++)$ and $(---)$ for $\kappa < \frac{1}{2}$; by $(++-)$, $(-++)$, $(--+)$, and $(+--)$ for $\kappa > \frac{1}{2}$; and by all these six configurations for $\kappa = \frac{1}{2}$. Patching this together, one obtains in particular the configurations of Fig. 11.

This yields two global ground states of $\mathcal{G}(\Phi)$ for $\kappa < \frac{1}{2}$ and four for $\kappa > \frac{1}{2}$. For $\kappa = \frac{1}{2}$ the number of ground states is infinite: the above calculations show that for $\kappa = \frac{1}{2}$ any configuration which does not contain a $(+-+)$ or $(-+-)$ interval, i.e., which does not contain separated plus

or minus spins, is a ground state (see Section 1.4). In the following, $\delta = \kappa - \frac{1}{2}$, and we consider δ close to 0. In most of the formulas we set $J_1 = 1$.

Following ref. 12, we consider now excitations obtained by flipping one spin. Vertical sections of five of them are pictured in Fig. 11 (horizontally instead of vertically); the others are obtained from these by applying symmetries of the interaction. The spin to be flipped is indicated by a caret; the labels o–τ are as in ref. 12; (E_i) indicates the order of the excitation, for J_0 large enough (note that, unlike here, in ref. 12 the order of an excitation is the number of the spins flipped); $f^o,\ldots, f^\tau$ are the corresponding one-point contributions to the free energy. Namely, defining

$$e_G(a) = \frac{1}{2} \sum_{\text{hor } B, a \in B} \Phi_B(G) + \frac{1}{3} \sum_{\text{ver } B, a \in B} \Phi_B(G)$$

and

$$f_{(\text{flipped spin at } a)}(G, a) = \beta e_G(a) - e^{-\beta H(\text{flip}_a G \mid G)}$$

one obtains, for example,

$$f^o = \beta(-2J_0 + \delta - \tfrac{1}{2}) - \exp - 4\beta(2J_0 + \tfrac{1}{2} - \delta) \simeq \beta\delta - zx^{1-2\delta}$$

where $\simeq$ means that we dropped the summand $-\beta(2J_0 - \frac{1}{2})$ common to all $f^o,\ldots, f^\tau$, and where $z = \exp(-8\beta J_0)$ $(= w^{q_\perp}$ of ref. 12, with $q_\perp = 4)$ and $x = \exp(-2\beta J_1)$. In the last column of Fig. 11 we give these free energies on the line $\delta = \delta_{(2)}^{\infty,3}$,

$$\beta\delta_{(2)}^{\infty,3} = -\tfrac{1}{2}z + \tfrac{3}{4}zx$$

of coexistence of the $\langle \infty \rangle$ and $\langle 3 \rangle$ phases (second-order perturbation expansion). $\delta_{(2)}$ is obtained by setting equal the perturbative free energies

$$f_{(2)}(\langle 3 \rangle) = \tfrac{1}{3}(2f^\sigma + f^\rho) \simeq -\tfrac{1}{3}\beta\delta - \tfrac{2}{3}z, \qquad f_{(2)}(\langle \infty \rangle) = f^o \simeq \beta\delta - zx^{1-2\delta} \tag{4.12}$$

$(e^{-\beta E_k} = zx^{k-1},\ k = 1, 2, 3, 4)$.

$$\begin{array}{lllll} o\text{:} & + + \hat{+} + + & E_2 & f^o \simeq \beta\delta - zx^{1-2\delta} \sim 0 \\ \pi\text{:} & + + \hat{+} + - & E_3 & f^\pi \simeq \tfrac{1}{3}\beta\delta - zx^2 \sim zx \\ \rho\text{:} & - + \hat{+} + - & E_4 & f^\rho \simeq \tfrac{1}{3}\beta\delta - zx^{3+2\delta} \sim zx \\ \sigma\text{:} & + + \hat{+} - - & E_1 & f^\sigma \simeq -\tfrac{1}{3}\beta\delta - z \sim -\tfrac{1}{2}zx \\ \tau\text{:} & - + \hat{+} - - & E_2 & f^\tau \simeq -\beta\delta - zx^{1+2\delta} \sim z \end{array}$$

Fig. 11. One-spin flips of the ANNNI model.

For each ground state G and a lattice point a we define now a second-order effective potential $f_{(2)}(G, a)$ as the contribution to the free energy coming from flipping spins at $a-\hat{z}$, a, and $a+\hat{z}$:

$$f_{(2)}(G, a) = \tfrac{1}{3}(f^{\alpha(a-\hat{z})} + f^{\alpha(a)} + f^{\alpha(a+\hat{z})}) \pmod{e^{-\beta E_3}}$$

where $\alpha(a) = o, \ldots, \tau$. Thus, $f_{(2)}(\langle \infty \rangle, a) = f^o$, $f_{(2)}(\langle 3 \rangle, a) = \frac{1}{3}(2f^\sigma + f^\rho)$, whereas, e.g., for $f_{(2)}(\langle 4 \rangle, a)$ one obtains alternatively two times $\frac{1}{3}(2f^\sigma + f^\pi)$ and two times $\frac{1}{3}(f^\sigma + 2f^\pi)$. Setting here $\delta = \delta_{(2)}^{\infty,3}$ and continuing these straightforward calculations, one obtains that $f_{(2)}(\langle \infty \rangle, a) = f_{(2)}(\langle 3 \rangle, a)$, and that if the restriction of a ground state G to the vertical interval of length 7 centered at a is not of class $\langle \infty \rangle$ or $\langle 3 \rangle$, then

$$f_{(2)}(G, a) - f_{(2)}(\langle \infty \rangle, a) \geqslant \tfrac{1}{6} zx$$

This implies that the six ground states of the class $\langle \infty \rangle$ and $\langle 3 \rangle$ are locally dominant in the sense of Section 2.4 (in second order) in a neighborhood of the line $\delta = \delta_{(2)}^{\infty,3}$. According to Theorem B and Remark 4 to that theorem, it follows that there exists a function $\beta \mapsto \delta^{\infty,3}(\beta)$ defined for β large enough such that (1) $\delta^{\infty,3} - \delta_{(2)}^{\infty,3}$ is of order zx^2 $(= e^{-\beta E_3})$ or higher; (2) on $\delta^{\infty,3}$ there are exactly eight pure phases, of classes $\langle \infty \rangle$ and $\langle 3 \rangle$; (3) there exists a constant $c > 0$ such that for $0 < \delta - \delta^{\infty,3} < czx$ $(= ce^{-\beta E_2})$ there are exactly six pure phases (of class $\langle 3 \rangle$) and for $0 < \delta^{\infty,3} - \delta < czx$ there are two pure phases (of class $\langle \infty \rangle$).

The same calculations imply that between the lines $\delta = \delta_{(2)}^{\infty,3}$ and $\delta = \delta_{(2)}^{3,2}$,

$$\beta \delta_{(2)}^{3,2} = z - \tfrac{3}{2} zx$$

the $\langle 3 \rangle$ ground states dominate in first order. Thus, for any $0 < a < 1$ there exists β_0 such that for $\beta > \beta_0$ and

$$\delta^{\infty,3}(\beta) < \delta < \frac{a}{\beta} z(\beta)$$

the only pure phases are those of class $\langle 3 \rangle$. [Extending these calculations to second order, we would obtain $\delta < \frac{1}{\beta}(z - azx)$, where $a > \frac{3}{2}$].

We could proceed in a similar manner to treat the $\langle 3 \rangle$–$\langle 23 \rangle$ coexistence line of ref. 12. We do not do this, however, since the entire phase diagram has already been treated rigorously in refs. 8–10 and 25. Moreover, using an abstract argument, one can in fact deduce the existence of an effective m-potential (2.5) from a somewhat strengthened version of "periodic calculations" of ref. 12 and thus confirm their results. The argument is similiar to that used in the fcc case to obtain the α–β coexistence line from the calculations of ref. 23. However, it is more involved, since here the ground states have more complicated symmetry

properties. One has to take into account that in "periodic calculations" many relations between the structural constants of ref. 12 are used, relations which are not satisfied by local ground states in general.

4.4. Stacked Models

Stacking yields three-dimensional models from two-dimensional ones, or, more generally, $(d+1)$-dimensional from d-dimensional ones. The intraplane part of the interaction of the stacked model in, say, horizontal planes is equal to that of the two-dimensional model, while the interplane part favors the same configuration of vertical neighbors. Thus, the ground states of a stacked model are obtained by repeating the same ground state of the two-dimensional model in each horizontal plane. Of course, one can also stack twice, starting from a one-dimensional model as in the case of the ANNNI model.

More precisely, stacking is defined as follows. Let Φ' be a potential on a d-dimensional lattice $\mathbb{L}'$. Then its (once) stacked version Φ (with parameter J) is the potential on the lattice $\mathbb{L} = \mathbb{L} \times \mathbb{Z}$ which is invariant under translations by the unit vector e in the z direction and such that for $M \subset \mathbb{L}' \times \{0\} \simeq \mathbb{L}'$, $\Phi_M(X) = \Phi'_M(X)$ and for $M = \{a, a+e\}$

$$\Phi_M(X) = -J\delta(X_a, X_{a+e}), \qquad J > 0$$

One has the following:

Suppose that the ground states of a two-dimensional system with the Hamiltonian H' extend uniquely from the boundaries. Then H has the retouch property and Condition $(\mathscr{L})$ is satisfied.

We skip the proof, which is obvious.

4.5. The Stacked Antiferromagnet

The Hamiltonian (1.7) can be rewritten as $\sum_P \Phi_P$, where the sum is over all the plaquettes of the lattice $\mathbb{Z}^2$ and

$$\Phi_P = \frac{1}{2} \sum_{\text{n.n. } a,b \in P} \sigma_a \sigma_b - \alpha \sum_{\text{n.n.n. } a,b \in P} \sigma_a \sigma_b - \frac{1}{4} h \sum_{a \in P} \sigma_a$$

Because of the flip symmetry, it is enough to consider $h \geqslant 0$. Minimizing Φ_P, one obtains Fig. 1, where in each region we indicate the (configurations of the) plaquette P with minimal Φ_P. We now restrict our attention to the strip $\mathbb{P}^1$ with base BC.

One can see by inspection that if each plaquette of a 4×4 square of the lattice is of type P^1 (i.e., a rotation of the plaquette P^1 of Fig. 1), then the configuration of the square is either of type α or of type β (Fig. 12); here by configuration of type α we understand a configuration obtained from that of Fig. 12 by a translation, rotation, or reflection.

Moreover, again by inspection, one can see that for the configuration α the vertical direction is soft while the horizontal is not; i.e., extension of the configuration to a ground state of a rectangle of height 4 is determined uniquely and it consists of plaquettes of the same type. For the β-plaquettes both the vertical and horizontal directions are soft. One can repeat now the reasoning applied to the Balanced Model in Section 4.1 and obtain both the structure of ground states, as described in Section 1.4, and the uniqueness of extensions of ground states from the boundaries, which by the preceding section implies the retouch property of the model (1.6). We proceed now to a description of low-energy excitations and a determination of the dominant ground states.

We call a (configuration X of a) plaquette P *excited* if $\Phi_P(X)$ is larger than for a ground-state plaquette, and we call it *second best* if there is no excited plaquette with a smaller Φ_P. A calculation yields that in the interior of the strip marked $\mathbb{P}^0$ on Fig. 1, the plaquette P^0 is second best. $\mathbb{P}^2_{nn}$ and $\mathbb{P}^2_{nnn}$ have similar properties, with P^2_{nn} and P^2_{nnn} replacing P^0. On the dashed lines the second best plaquettes are those of the bordering regions, while at the point A all three types of plaquettes are second best.

Now, it is easy to see that any excitation has at least four excited plaquettes. Therefore, for $J>0$ lowest order excitations are obtained by flipping one spin, and the parts $\mathbb{P}^2_{nn}$ and $\mathbb{P}^2_{nnn}$ of Fig. 12 show that in the regions $\mathbb{P}^2_{nn}$ and $\mathbb{P}^2_{nnn}$ of Fig. 1 the four equivalent ground states containing

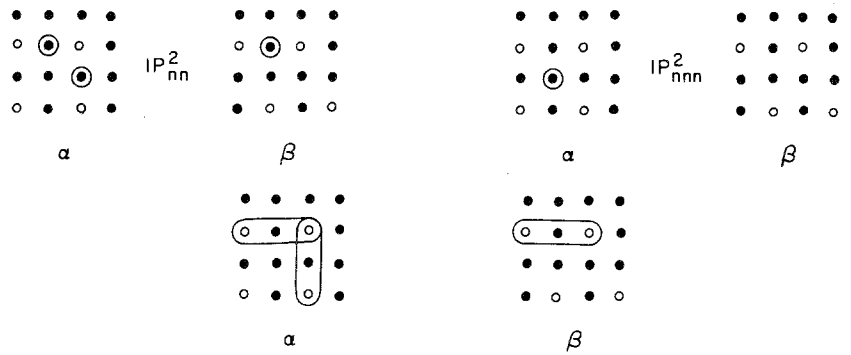

Fig. 12. Ground states of the 4×4 squares of the stacked antiferromagnet (1.6) in the strip $\mathbb{P}^1$. (●) plus spins; (○) minus spins. Excitations yielding domination are obtained by flipping the circled spins.

only the α-squares are dominant. For, consider first the (open) strip $\mathbb{P}^2_{nn}$. Here there are two ways of flipping an interior spin of an α-square which yield four P^2_{nn} plaquettes, whereas there is only one such way for the β-square. It is easy to obtain from this an effective potential showing local domination of the α-ground states, as in the preceding sections. Similar considerations yield one way of producing four P^2_{nnn} plaquettes for an α-square and none for the β-square, which shows that again the four α-ground states are dominant in order 1.

In the region $\mathbb{P}^0$ the situation is more complicated, since there is one excitation of first order both for the α- and β-squares. For $J>0$, in the interior of $\mathbb{P}^0$ but *close* to its border with the $\mathbb{P}^2_{nn}$ and $\mathbb{P}^2_{nnn}$, the second-order excitations are obtained by those flippings which produce four P^2_{nn} or P^2_{nnn} plaquettes, respectively. Thus, in this boundary region the α-ground states are dominant in second order. If J is small enough, then close to the boundary of $\mathbb{P}^0$ second-order excitations are obtained by flipping three horizontal or vertical spins, as indicated in Fig. 12. Then again the α-ground states are dominant in order 2. And again it is easy to write down an effective m-potential demonstrating local domination here. We skip this.

This ends the proof of the fact that the four equivalent ground states of $\mathcal{G}^*$ of Section 1.4 are dominant for (α, h) in the strip $\mathbb{P}^0$, and that therefore by Theorem A the system has there exactly four pure phases at low temperatures.

5. CONCLUDING REMARKS

We discuss now several classes of models with an infinite number of periodic ground states, most of which cannot be treated with the methods of this paper, and some of which constitute interesting open problems.

(a) *Models of interfaces.* An infinite set of ground states may also occur in systems with an infinite spin space, e.g., various SOS models of interfaces, and interacting surfaces. These systems may have quite a rich phase diagram, such as wetting and layering transitions, which can be analyzed with an extension of Pirogov–Sinai theory related to the one developed here[2] (see also ref. 1 for a different approach to these models).

(b) *Ferromagnetic models.* Here the Hamiltonian has a large group of symmetries, which acts transitively on the set of ground states. Using these symmetries, a complete analysis of the low-temperature phase diagram is possible, irrespective of the ground-state degeneracy.[20,26,32]

(c) *Models with nonzero ground-state entropy.* The first problem is to understand what happens at zero temperature, where the set of ground

states forms an ordinary statistical mechanical system in itself. If this is done, then in some cases, the method developed here may be used, in combination with those of ref. 11, to give some information on the low-temperature phase diagram (see ref. 4 for an elaboration on this point). Recently a special situation has been treated in ref. 42.

(d) *Regularity.* There seems to exist a relation between the degree of ground-state degeneracy of a model and its regularity. Indeed, consider a three-dimensional model and let us count the number of ground-state configurations in a cube of side L. In all the regular models that we have encountered, we found that this number grows like c^L. It may grow like c^{L^2}, as in the staked antiferromagnet,[5] but this model is not regular, while having zero ground-state entropy (nonzero ground-state entropy corresponds to a growth rate of c^{L^3}). On the other hand, in two dimensions the only regular models that we have found are those with a finite number of ground states. Models with a number of ground states growing like c^L have zero ground-state entropy, but are not regular.

(e) *Nonregular models with zero ground-state entropy.* This is probably the most interesting class of open problems for which progress seems to be possible. It includes the two-dimensional Balanced, ANNNI, and antiferromagnet (1.6) models, the Heilmann–Lieb models of liquid crystals,[19] the Domino model,[35] the stacked antiferromagnet of ref. 5, or Pecherski's model[32] (a model with a finite set of periodic ground states which does not satisfy the Peierls condition). All these models have zero ground-state entropy. There does not exist a general method, even at a formal level, to analyze these diagrams, although several examples have been studied (two-dimensional ANNNI model,[13,36] domino model,[19,34,35] and partially understood.

(f) *Existence and estimates of the surface tension.* In many models, phase transitions may be discussed equivalently in terms of coexistence of phases or in terms of surface tension (see, e.g., ref. 21). As pointed out by Binder (private communication via Joel Lebowitz), the surface tension behaves quite differently in systems with a finite and with an infinite number of ground states. In the latter case the surface tension may go exponentially to zero as β tends to infinity, while it approaches a nonzero value in the same limit in the Ising model and in all the models covered by the PS theory.[40] This is easy to understand: when there is an infinite number of ground states, it may happen that an interface between two phases does not cost any energy, but costs only a free energy of low-energy excitations. This is indeed the basic mechanism studied in this paper. The surface tension measures the cost, in free energy, of an interface between two phases. Since the free energy of these low-energy excitations is exponen-

tially small at low temperatures, so is the surface tension. We give now a more precise formulation of the result.

A standard definition of the surface tension is a follows. Let $\Lambda = \Lambda_{L,M}$ be a rectangular box of size $M \times L^{d-1}$ and let $Z(\Lambda \mid G, G')$ be the partition function with boundary conditions G on the upper half of Λ and G' on the bottom half. Then the surface tension between the G and G' phases is

$$\tau = \lim_{L,M \to \infty} \tau_{L,M}$$

where

$$-\tau_{L,M} = L^{-(d-1)} \log \frac{Z(\Lambda \mid G.\, G')}{[Z(\Lambda \mid G)\, Z(\Lambda \mid G')]^{1/2}}$$

(assuming the limits exist). Now, following the proof of the Peierls bound (3.8), one can derive the lower bound

$$\tau_{L,M} \geqslant e^{-O(\beta)}$$

uniformly in L and M.

APPENDIX: PROOF OF UNIQUENESS

We prove here the uniqueness part of Theorems A and B (see the beginning of Section 3.5). We consider the framework of Theorem B. Thus, for each ground state G of $\mathscr{G}^*$ we have a τ-functional F_G and a parameter b_G satisfying (3.35). Let

$$\mathscr{G}^{**} = \mathscr{G}^{**}(\mu) = \{G \in \mathscr{G}^*: b_G = 0\}$$

The ground states in $\mathscr{G}^{**}$ will be called *stable* and the remaining ground states of $\mathscr{G}^*$ will be called *unstable*. We use the same terminology for the corresponding restricted ensembles. We shall prove that the only pure phases are small perturbations of the stable ground states, as in the Pirogov–Sinai theory.[29,30] We explain first the main ideas of the proof.

By (3.35a), for any boundary conditions Y in the restricted ensemble of G, the partition function $Z^0(\Lambda \mid Y)$ is equal to that of a contour model with parameter b_G. Equation (3.29) implies that large contours are favored in an ensemble defined by a contour model with a nonzero parameter. This expresses the intuitive notion that if one puts boundary conditions belonging to an unstable ground state on a large enough box, then a large contour enclosing most of the box and whose interior boundary belongs to a stable ground state will appear. However, it is not obvious how to turn this

idea into a proof. Part of the difficulty is that b_G may be arbitrarily small. However, by going to a large enough scale, one should be able to see that the unstable ground states are suppressed. Our main idea is to define new "large-scale" contours which will be regions of the lattice where the configuration either contains large contours (of the type defined in Section 3.1) or belongs to an unstable restricted ensemble. The "large scale" will depend on the size of the boxes Λ along which the thermodynamic limit is taken. Thus, the size of these boxes will play a role similar to the one played by β in Section 3 [compare (A.16) with (2.9)]. This will allow us to prove that under *any* boundary conditions, the total fraction of the volume of Λ where the configuration coincides with an unstable restricted ensemble is negligible when $\Lambda \to \infty$. This and standard arguments[15,4] will imply that all extremal periodic Gibbs states are small perturbations of stable ground states.

To fix the notation, let $d=3$ and let Λ be a large box centered at the origin, with $|\Lambda| = (NL)^3$, where L is fixed as in Section 3.1, and N goes to infinity with Λ. We cover Λ with large cubes (called *N-cubes*),

$$B_a = B_0 + \frac{N^{1/2}L}{2}\, a, \qquad a \in \mathbb{Z}^3, \quad \text{where} \quad |B_0| = L^3 N^{3/2}$$

Observe that the size of B_0 depends on Λ ($|B_0| \simeq |\Lambda|^{1/2}$), and that the number of N-cubes in Λ is of order $N^{3/2}$.

A contour (as defined in Section 3.1) is *small* if $|\Gamma|_L \leqslant N$, and *large* otherwise.

We remark that the probability in a contour model with τ large of having at least *one* large contour in Λ is very small, i.e., of order

$$|\Lambda|\,(ce^{-\tau})^N \leqslant e^{-\tau N/2} \tag{A.1}$$

for N large [since $|\Lambda| = (LN)^3$]. This follows immediately from the definition (3.23) of a τ-functional. Now we define our new "contours," *N-contours*, as unions of N-cubes: An N-cube B is *regular* for a given configuration if:

1. All contours intersecting B are small, and
2. In the components of the complement of the contours in B [i.e., in $B \setminus \bigcup_{\Gamma \cap B \neq \varnothing} \theta(\Gamma)$], the configuration belongs to a *stable* restricted ensemble. Thus, we have again two types of irregular cubes, those of type I (with an unstable restricted ensemble) and those of type II (with a large contour).

Let Q be the union of all irregular N-cubes. This is not necessarily a connected set, since we do not perform here the decomposition into

connected components, contrary to the case of usual contours. Our *main estimate* follows:

There exists $c>0$ such that for all N large enough,

$$P_\Lambda(Q \mid Y) \leqslant e^{-cN|Q|_N} \qquad \text{for all} \quad Y \in \mathscr{X}^{G,E} \text{ and } G \in \mathscr{G}^{**} \tag{A.2}$$

where $|Q|_N$ is the number of N-cubes in Q,

$$P_\Lambda(Q \mid Y) = Z(\Lambda, Q \mid Y)/Z(\Lambda \mid Y) \tag{A.3}$$

is the probability of Q in the Gibbs state in Λ with boundary conditions Y, and $Z(\cdot \mid Y)$ is as at the beginning of Section 3.2.

(Note that $|Q| \simeq L^3N^{3/2}|Q|_N$.) (A.2) implies that for *any* boundary condition X,

$$P_\Lambda(Q \mid X) \leqslant \exp[c'\beta(LN)^2 - cN|Q|_N] \tag{A.4}$$

because P is a ratio of partition functions, and changing boundary conditions affects partition functions at most by a factor of $\exp[O(\beta)(LN)^2]$. [The $Z(\cdot \mid Y)$ at the beginning of Section 3.2 is defined for boundary conditions in a restricted ensemble, and so is P in (A.3). In the general case of (A.4), P is defined in the standard way.]

Using (A.4), we may bound, for any boundary condition X,

$$\begin{aligned} &P_\Lambda(|Q|_N \geqslant N^{5/4} \mid X) \\ &\quad \leqslant \exp[O(N^{3/2}) + O(\beta)(LN)^2 - cN \cdot N^{5/4}] \leqslant \exp[-O(N^{9/4})] \end{aligned} \tag{A.5}$$

since there are $e^{O(N^{3/2})}$ possible families Q of N-cubes in Λ. The condition (A.5) tends to zero as $N \to \infty$. So, we may as well condition onto the event that $|Q|_N \leqslant N^{5/4}$. But then $|Q|$ has a bound of order $(L\sqrt{N})^3N^{5/4} = L^3N^{11/4}$, which is much smaller than $|\Lambda| = L^3N^3$. Therefore $|\Lambda \backslash Q|/|\Lambda| \to 1$ as Λ tends to infinity and all but a negligible part of Λ is made of regions where the configuration consists of a stable restricted ensemble with small contours. Moreover, as we discuss below, there exists a convergent expansion in those regions. These facts and standard arguments,[15,4] will conclude our proof.

In order to prove (A.2), we introduce the *small contour partition function*:

$$Z_S^0(\Lambda \mid Y) = \sum_X^S e^{-\beta H(X)}/Z_E(\Lambda \mid Y) \tag{A.6}$$

where $Y \in \mathscr{X}^{G,E}$ for some $G \in \mathscr{G}^*$ and the sum extends over those configurations defining $Z(\Lambda \mid Y)$ which have small contours only.

Using (3.9) and (3.35a), we have

$$Z_S^0(\Lambda \mid Y) = \sum_{\omega}^{S} e^{b_G v(\omega) - W_\Lambda(\omega \mid Y)} \prod_{\Gamma \in \omega} Z(\Gamma; F) \tag{A.7}$$

where the sum runs over families of small exterior contours (note that all contours are small if the exterior contours are small). $b_G = 0$ for $G \in \mathscr{G}^{**}$, and one has a convergent cluster expansion in the contour model, as discussed in Section 3.4. Thus, one has a formula analogous to (2.2):

$$\log Z_S^0(\Lambda \mid Y) = \sum_{\vartheta} \phi^T(\vartheta, \Lambda, Y) \tag{A.8}$$

if $Y \in \mathscr{X}^{G,E}$. Here the ϑ's are m.f. on the polymers of the contour model, with only small contours present. Moreover,

$$\log Z_S^0(\Lambda \mid Y) = s(F_G)|\Lambda| + O(e^{-\tau})|\partial\Lambda|_L \tag{A.9}$$

This is similar to (3.26), and follows from the fact that the restriction to small contours produces only a negligible error [see (A.1)] for Λ large.

The proof of (A.2) is similar to that of (3.10). The role of the fundamental estimation (2.9) is played here by the following.

Lemma. There exists $c > 0$ such that if $b_G \neq 0$, for any large enough N and any subset Λ' of Λ with $|\Lambda'| \geqslant N$,

$$Z_S^0(\Lambda' \mid Y) \leqslant \exp([s(F_G) + b_G]|\Lambda'| - cN^{-1/2}|\Lambda'|_L + \{\exp[-O(\beta)]\}|\partial\Lambda'|_L) \tag{A.10}$$

Now, using the lemma, we finish the proof of (A.2). Comparing the numerator and denominator in (A.3), one obtains a bound with a factor $e^{-O(\rho)|\Gamma|_L}$ for each large contour and, using the lemma, a factor

$$\exp(-cN^{-1/2}|B_i|_L) = \exp(-c'N)$$

for each irregular N-cube of type I.

More precisely, we shall show that there exists $c > 0$ such that

$$P_\Lambda(Q \mid Y) \leqslant \sum_{\omega}^{l} \prod_{\Gamma \in \omega} \exp(-\psi_\Lambda(\Gamma \mid Y) + \{\exp[-O(\beta)]\}|\Gamma|_L - cN|\tilde{\Lambda}(\omega)|_N) \tag{A.11}$$

where the sum is over families ω of large contours with support contained in Q. To define $\tilde{\Lambda}(\omega)$, let

$$\Lambda \backslash \mathrm{supp}(\omega) = \bigcup_{G \in \mathscr{G}^*} \Lambda_G(\omega)$$

be the decomposition of the complement of the support of ω into connected regions having as boundary conditions a configuration of $\bigcup_{G \in \mathscr{G}^*} \mathscr{X}^{G,E}$. Then

$$\tilde{\Lambda}(\omega) = \bigcup_{G \notin \mathscr{G}^{**}} \Lambda_G(\omega)$$

is the union of the regions having unstable boundary conditions. Then, using (A.11) instead of (3.13), we can derive (A.2) by arguments similar to the proof of (3.10) [using now (3.10) and (A.1) to control the sum over the large contours].

To prove (A.11), we write in (A.3)

$$Z(\Lambda, Q \mid Y) = \sum_{\omega}{}' \prod_{\Gamma \in \omega} e^{-\beta H(\Gamma)} \prod_{G \in \mathscr{G}^*} Z_S^0(\Lambda_G(\omega) \mid Y_G) Z_E(\Lambda_G(\omega) \mid Y_G) \quad \text{(A.12)}$$

where the sum is over large contours in Q and $Y_G \in \mathscr{X}^{G,E}$ are the boundary conditions on $\Lambda_G(\omega)$ determined by the contours in ω. Of course, Y_G coincides with the Y of (A.3) in the complement of Λ. [(A.12) follows from the definition (A.6) of the small contour partition function.] By restricting the summation in the denominator $Z(\Lambda \mid Y)$ of P to configurations with small contours only, we bound it from below by

$$Z_S^0(\Lambda \mid Y) Z_E(\Lambda \mid Y) \quad \text{(A.13)}$$

We estimate now the small-contour partition functions. In (A.12), we use the expansion (A.8), (A.9) for the stable G's and we use it also in (A.13) (where G is stable by assumption). For $G \notin \mathscr{G}^{**}$ in (A.12) we use the upper bound (A.10). Now comparing the numerator and denominator, (A.12) and (A.13), we observe that we get from (A.9) and (A.10) a term

$$\exp\left\{\sum_G [s(F_G) + b_G - s(F_{G(Y)})] |\Lambda_G(\omega)| - cN^{-1/2} |\tilde{\Lambda}(\omega)|_L\right\} \quad \text{(A.14)}$$

in the numerator plus boundary terms. Those coming from the boundary of Λ cancel between the numerator and the denominator of (A.3) because we have the same boundary conditions there, or contribute to the $e^{-O(\beta)|\Gamma|_L}$ term in (A.11), like the other boundary terms [see (A.9)]. Now we use (3.35b), which we write as

$$s(F_G) + b_G - s(F_{G(Y)}) = f_E(G) - f_E(G(Y))$$

(since $b_{G(Y)}=0$), and insert this in (A.14). Going back to (A.12), we have

$$P_\Lambda(q\,|\,y)\leqslant Z_E(\Lambda\,|\,Y)^{-1}\sum_{\omega}^{I}\prod_{\Gamma\in\omega}\exp(-\beta H(\Gamma)+\{\exp[-O(\beta)]\}\,|\Gamma|_L)$$
$$\times\prod_{G\in\mathscr{G}^*}Z_E(\Lambda_G(\omega)\,|\,Y_G)$$
$$\times\exp\left\{\sum_{G\in\mathscr{G}^*}[f_E(G)-f_E(G(Y))]\,|\Lambda_G(\omega)|-cN^{-1/2}|\tilde{\Lambda}(\omega)|_L\right\} \tag{A.15}$$

Now, using the convergent expansion of the restricted partition function, we see that

$$-\beta\sum_{\Gamma\in\omega}H(\Gamma)+\sum_{G\in\mathscr{G}^*}\{\log Z_E(\Lambda_G(\omega)\,|\,Y_G)+[f_E(G)-f_E(G(Y))]\,|\Lambda_G(\omega)|\}$$
$$-\log Z_E(\Lambda\,|\,Y)$$
$$\leqslant-\sum_{\Gamma\in\omega}\psi_\Lambda(\Gamma\,|\,Y)+e^{-O(\beta)}|\Gamma|_L \tag{A.16}$$

To prove (A.16), go back to the definition (3.5) of $\psi_\Lambda(\Gamma\,|\,Y)$, which now can be rewritten as

$$\psi_\Lambda(\Gamma\,|\,Y)=\beta H(\Gamma)-\sum_{G\in\mathscr{G}^*}\{\log Z_E(\Lambda_G(\Gamma)\,|\,\Gamma,\,Y)$$
$$+[f_E(G)-f_E(G(Y))]\,|\Lambda_G(\Gamma)|\}+\log Z_E(\Lambda\,|\,Y)$$

where $\Lambda_G(\Gamma)=\Lambda_G(\omega)$ for $\omega=\{\Gamma\}$, and in the second term on the rhs we have boundary conditions determined by Γ and Y. Then we see that the difference between the lhs of (A.16) and the first term on its rhs is similar to $U_\Lambda(\omega\,|\,Y)$ in (3.3) and, using the expansion in the restricted ensembles, can be bounded by $e^{-O(\beta)}|\Gamma|_L$, i.e., the second term in the rhs of (A.16). Inserting (A.16) into (A.15) proves (A.11). Using (A.11) and fact that $|\tilde{\Lambda}(\omega)|_L\simeq N^{3/2}|\tilde{\Lambda}(\omega)|_N$, we finish the proof of (A.2). It remains to prove the lemma.

Proof of Lemma. Using (3.7) and (3.35a), we have

$$Z_S^0(\Lambda'\,|\,Y)=\sum_{\omega}^{S}e^{bv(\omega)-W_\Lambda(\omega\,|\,Y)}\prod_{\Gamma\in\omega}Z(\Gamma;F) \tag{A.17}$$

where $F=F_{G(Y)}$ and the sum runs over families of small exterior contours. Let us split this sum into $\sum_\omega^1+\sum_\omega^2$, where the first sum runs over ω such that $v(\omega)\leqslant|\Lambda'|/2$ and the second over ω with $v(\omega)>|\Lambda'|/2$. We shall

bound each sum by the rhs of (A.10). Then the lemma follows with possibly a smaller c. The first sum is easy:

$$bv(\omega) \leqslant b|\Lambda'| - \frac{b}{2}|\Lambda'| \qquad \text{and} \qquad \frac{b}{2} \geqslant \frac{c}{N^{1/2}} \tag{A.18}$$

for N large enough. Now the upper bound

$$\sum_{\omega}^{S} \{\exp[-W_{\Lambda}(\omega|Y)]\} \prod_{\Gamma \in \omega} Z(\Gamma; F)$$

$$\leqslant Z(\Lambda'; F|Y) \leqslant \exp(s(F_G)|\Lambda'| + \{\exp[-O(\beta)]\}|\partial\Lambda'|_L) \tag{A.19}$$

follows from (3.26) and (3.27).

Consider now $\sum_{\omega}^{2}$ and introduce

$$g(\Gamma) \equiv \exp\left(\frac{v(\Gamma)}{L^3 N^{1/2}}\right)$$

By the isoperimetric inequality and $|\Gamma|_L \leqslant N$ (small contours) we have

$$\frac{v(\Gamma)}{L^3 N^{1/2}} \leqslant c \frac{|\Gamma|_L^{3/2}}{N^{1/2}} \leqslant c|\Gamma|_L \tag{A.20}$$

By a Tchebyshev inequality,

$$\sum_{\omega}^{2} \{\exp[bv(\omega) - W_{\Lambda}(\omega|Y)]\} \prod_{\Gamma \in \omega} Z(\Gamma; F)$$

$$\leqslant \exp\left(b|\Lambda'| - \frac{c|\Lambda'|_L}{N^{1/2}}\right) \sum_{\omega} \{\exp[-W_{\Lambda}(\omega|Y)]\} \prod_{\Gamma \in \omega} Z(\Gamma; F, g) \tag{A.21}$$

where in $Z(\Gamma; F, g)$ we replaced $e^{-F(\Gamma)}$ by $e^{-F(\Gamma)} g(\Gamma)$ $[g(\Gamma) \geqslant 1!]$, and where we used $v(\omega) > |\Lambda'|/2$ to obtain the first exponential. We write the sum $\sum_{\omega}$ in (A.21) as

$$\langle g \rangle(\Lambda'; F|Y)\, Z(\Lambda'; F|Y)$$

We shall show that

$$\langle g \rangle(\Lambda'; F|Y) \leqslant \exp\left(\frac{O(e^{-\tau/2})|\Lambda'|_L}{N^{1/2}}\right) \tag{A.22}$$

Then, using the bound (A.19) on $Z(\Lambda'; F|Y)$, the proof of lemma will be finished [$O(e^{-\tau/2}) \ll c/2$ for β large]. To prove (A.22), we observe that, due to (A.20), $g(\Gamma)$ is a "small correction" to $e^{-F(\Gamma)}$ in the sense that

$$\sum_{\mathrm{supp}(\Gamma)=M} |g(\Gamma)-1| e^{-F(\Gamma)} \leqslant \frac{e^{-\tau|M|_L/2}}{N^{1/2}} \tag{A.23}$$

[To get (A.23), use $|e^x - 1| \leqslant |x| e^{|x|}$, (A.20), and the fact that τ is large.] Coming back to $\langle g \rangle(\Lambda'; F|Y)$, we consider it as a ratio of partition functions, for contour models defined by $\tau/2$-functionals: we have $g(\Gamma)e^{-F(\Gamma)}$ in the numerator and $e^{-F(\Gamma)}$ in the denominator [use (A.20) again to prove that $\Gamma \mapsto g(\Gamma)e^{-F(\Gamma)}$ is a $\tau/2$-functional]. Use the convergent expansion (A.8) (without the restriction to small contours) on both terms of that ratio, which gives

$$\langle g \rangle(\Lambda'; F|Y) = \exp\left\{\sum_{\vartheta} \phi^T(\vartheta, Y)[g(\vartheta)-1]\right\}$$

To get this last formula, we use the fact that $g(\cdot)$ is a multiplicative functions on contours,

$$g(\omega) = \prod_{\Gamma \in \omega} g(\Gamma), \qquad g(\vartheta) = \prod_{\Gamma} g(\Gamma)^{\vartheta(\Gamma)}$$

Now, using

$$|g(\vartheta)-1| \leqslant \sum_{\Gamma} \theta(\Gamma)\, |g(\Gamma)-1|\, g(\vartheta'_\Gamma)$$

where $\vartheta'_\Gamma(\Gamma') = \theta(\Gamma')$ for $\Gamma' \neq \Gamma$, and $\vartheta'_\Gamma(\Gamma) = 0$, (A.23), and the usual properties of $\phi^T(\vartheta, Y)$, we obtain (A.22).

ACKNOWLEDGMENTS

J. B. would like to thank Ya. G. Sinai for conversations related to the subject of this paper. Both authors are grateful to Joel Lebowitz for encouragement and warm hospitality at Rutgers University. This work has been supported in part by NSF grant PHY-8515288 A01 (J. B. at Princeton University), MCS-8301709, and DMS-8700580 (J. S. at Virginia Tech), and DMR-8612369 (both authors at Rutgers University).

REFERENCES

1. A. G. Basuev, *Theor. Math. Phys.* **58**:171 (1984).
2. J. Bricmont, A. El Mellouki, and J. Fröhlich, *J. Stat. Phys.* **42**:743 (1986).
3. J. Bricmont, K. Kuroda, and J. L. Lebowitz, *Commun. Math. Phys.* **101**:501 (1985).
4. J. Bricmont and J. Slawny, in *Statistical Mechanics and Field Theory: Mathematical Aspects* (Proc. Int. Conf., Groningen, August 1985), T. C. Dorlas, N. M. Hugenholtz, and M. Winnik, eds. (Springer-Verlag, 1986).
5. S. N. Coppersmith, *Phys. Rev. B* **32**:1584 (1985).
6. A. Danielian, *Phys. Rev. Lett.* **16**:670 (1961).
7. A. Danielian, *Phys. Rev. A* **133**:1399 (1964).
8. E. L. Dinaburg and A. E. Mazel, in *Proceedings 8th International Congress on Mathematical Physics*, M. Mebkhout and R. Sénéor, eds. (World Scientific, Singapore, 1987).
9. E. L. Dinaburg, A. E. Mazel, and Ya. G. Sinai, in *Mathematical Physics Reviews* (Sov. Sci. Rev. Sect. C), Vol. 6, S. P. Novikov, ed. (Gordon and Breach, New York, 1986).
10. E. L. Dinaburg and Ya. G. Sinai, *Commun. Math. Phys.* **98**:119 (1985).
11. R. L. Dobrushin and S. B. Shlosman, in *Mathemaical Physics Reviews* (Sov. Sci. Rev. Sect. C), Vol. 5, S. P. Novikov, ed. (Gordon and Breach, New York, 1986).
12. M. E. Fisher and W. Selke, *Phil. Trans. R. Soc.* **302**:1 (1981).
13. M. E. Fisher and W. Selke, *Phys. Rev. B* **20**:257 (1979).
14. J. Fröhlich, R. B. Israel, E. H. Lieb, and B. Simon, *J. Stat. Phys.* **22**:297 (1980).
15. G. Gallavotti and S. Miracle-Sole, *Phys. Rev. B* **5**:2555 (1972).
16. K. Gawedzki, K. Kotecky, and A. Kupiainen, *J. Stat. Phys.* **47**:701 (1987).
17. V. M. Gerzik, *Izv. Akad. Nauk SSSR Ser. Mat.* **40**:448 (1976).
18. V. M. Gerzik and R. L. Dobrushin, *Funkts. Anal. Prilozh.* **8**:12 (1974).
19. O. J. Heilmann and E. H. Lieb, *J. Stat. Phys.* **20**:679 (1979).
20. W. Holsztynski and J. Slawny, *Commun. Math. Phys.* **66**:147 (1979).
21. J. L. Lebowitz and C. Pfister, *Phys. Rev. Lett.* **46**:1031 (1981).
22. J. L. Lebowitz, M. K. Phani, and D. F. Styer, *J. Stat. Phys.* **38**:413 (1985).
23. N. D. Mackenzie and A. P. Young, *J. Phys. C* **14**:3927 (1981).
24. D. G. Martirosyan, *Izv. Akad. Nauk Arm. SSR, Ser. Mat.* **14**:21 (1979) [English transl. *Sov. J. Contemp. Math. Anal.* **14**:19 (1979)].
25. A. E. Mazel, *Theor. Math. Phys.* **66**:832 (1986).
26. J. Miekisz, *Commun. Math. Phys.* **109**:353 (1987); *J. Phys. A*, to appear.
27. Y. M. Park, Extension of Pirogov–Sinai theory of phase transitions to infinite range interactions I: Cluster expansion; II: Phase diagram, *Commun. Math. Phys.* **114**:187 (1988) and **114**:219 (1988).
28. S. A. Pirogov, *Teor. Mat. Fiz.* **66**:218 (1986).
29. S. A. Pirogov and Ya. G. Sinai, *Teor. Mat. Fiz.* **25**:358 (1975); **26**:61 (1976).
30. Ya. G. Sinai, *Theory od Phase Transitions: Rigorous Results* (Pergamon Press, Oxford, 1982).
31. J. Slawny, *J. Stat. Phys.* **20**:711 (1979).
32. J. Slawny, in *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, London, 1987).
33. D. F. Styer, *Phys. Rev. B* **32**:393 (1985).
34. J. Villain and P. Bak, *J. Phys.* (Paris) **42**:657 (1981).
35. J. Villain, J.-P. Bidaux, and R. Conte, *J. Phys.* (Paris) **41**:1263 (1980).
36. J. Villain and M. B. Gordon, *J. Phys. C* **13**:3117 (1980).
37. F. Wegner, *J. Math. Phys.* **12**:2259 (1971).
38. M. Zahradnik, *Commun. Math. Phys.* **93**:559 (1984).

39. M. Zahradnik, in *Proceedings 8th International Congress on Mathematical Physics*, M. Mebkhout and R. Séneor, eds. (World Scientific, Singapore, 1987).
40. J. Bricmont, K. Kuroda, and J. L. Lebowitz, *J. Stat. Phys.* **33**:59 (1983).
41. C. Gruber, A. Hinterman, and D. Merlini, *Group Analysis of Classical Lattice Systems* (Springer-Verlag, 1977).
42. C. Gruber and A. Sütő, *J. Stat. Phys.*, to appear.
43. R. L. Dobrushin and M. Zahradnik, in *Mathematical Problems of Statistical Physics and Dynamics*, R. L. Dobrushin, ed. (Reidel, 1985).
44. J. Z. Imbrie, *Commun. Math. Phys.* **82**:261, 305 (1981).
45. C. Domb and M. S. Green, eds., *Phase Transitions and Critical Phenomena*, Vol. 3 (Academic Press, 1974).
46. A. G. Basuev, *Theor. Math. Phys.* **64**:716 (1985); **72**:861 (1987).